

Killing Initial Data on spacelike conformal boundaries*

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Abstract

We analyze Killing Initial Data on Cauchy surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. As an application, we derive the KID equations on a spacelike \mathcal{S}^- .

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Contents

1	Introduction	2
2	Setting	3
2.1	Conformal field equations	3
2.2	Gauge freedom	4
2.2.1	Conformal factor	4
2.2.2	Coordinates	5
2.2.3	Realization of the gauge scheme	6
2.3	Constraint equations in the $(R = 0, W^\lambda = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu})$ -wave map gauge	6
2.4	Well-posedness of the Cauchy problem on a spacelike \mathcal{S}^-	9
3	KID equations	9
3.1	Unphysical Killing equations	9
3.2	KID equations on a Cauchy surface	10
3.3	A special case: $\Theta = 1$	11
3.4	A stronger version of Theorem 3.1	12
3.5	The (proper) KID equations	13

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4	KID equations on a spacelike \mathcal{I}^-	14
4.1	Derivation of the (reduced) KID equations	14
4.2	Properties of the reduced KID equations	16
4.3	Some special cases	17
4.3.1	Compact initial manifolds	17
4.3.2	Maximally symmetric space-times	17
4.3.3	Non-existence of stationary space-times	18
A	Equivalence between the CWE and the MCFE	19
A.1	Conformal wave equations (CWE)	19
A.2	An intermediate result	19
A.3	Applicability of Theorem A.1 on \mathcal{I}^-	20
A.3.1	Vanishing of \overline{H} , $\overline{\nabla H}$ and $\overline{\nabla \nabla H}$	21
A.3.2	Vanishing of $\overline{\zeta}$ and $\overline{\nabla \zeta}$	22
A.3.3	Validity of the MCFE (2.1)-(2.4) and their transverse derivatives on \mathcal{I}^-	23
A.3.4	Vanishing of $\overline{W}_{\mu\nu\sigma}{}^\rho - \Theta \overline{d}_{\mu\nu\sigma}{}^\rho$ and $\overline{\nabla}_0(W_{\mu\nu\sigma}{}^\rho - \Theta d_{\mu\nu\sigma}{}^\rho)$	25

1 Introduction

Symmetries are of utmost importance in physics, and so is the construction of space-times (\mathcal{M}, \tilde{g}) satisfying Einstein's field equations in general relativity which possess k -parameter groups of isometries, $1 \leq k \leq 10$ when $\dim \mathcal{M} = 4$, generated by so-called Killing vector fields. Indeed, such space-times can be systematically constructed in terms of an initial value problem when the usual constraint equations, which are required to be fulfilled by appropriately prescribed initial data, are supplemented by certain additional equations, the Killing Initial Data (KID) equations.

The KID equations have been derived on spacelike as well as characteristic initial surfaces (cf. [1, 3] and references therein). In [10] the same issue was analyzed for characteristic surfaces in conformally rescaled vacuum space-times satisfying Friedrich's conformal field equations. In particular, for vanishing cosmological constant, the KID equations on a light-cone with vertex at past time-like infinity have been derived there. The aim of this work is to carry out the corresponding analysis on spacelike hypersurfaces in conformally rescaled vacuum space-times. As a special case we shall derive the KID equations on \mathcal{I}^- supposing that the cosmological constant is positive so that \mathcal{I}^- is a spacelike hypersurface.

In Section 2 we recall the conformal field equations, discuss their gauge freedom and derive the constraint equations induced on \mathcal{I}^- . Well-posedness of the Cauchy problem for the conformal field equations with data on \mathcal{I}^- was shown in [4], we shall provide an alternative proof based on results proved in Appendix A by using a system of wave equations.

The “unphysical Killing equations”, introduced in [10] replace, and are in fact equivalent to, the original-space-time Killing equations in the unphysical space-time. Employing results in [10] we derive in Section 3 necessary-and-sufficient conditions on a spacelike hypersurface in a space-time satisfying the conformal field equations which guarantee existence of a vector field fulfilling these equations (cf. Theorem 3.3). Similar to the proceeding in [3, 10] we first

derive an intermediate result, Theorem 3.1, with a couple of additional hypotheses, which then are shown to be automatically satisfied.

In Section 4 we apply Theorem 3.3 to the special case where the spacelike hypersurface is \mathcal{S}^- . We shall see that some of the KID equations determine a set of candidate fields on \mathcal{S}^- . Whether or not these fields extend to vector fields satisfying the unphysical Killing equations depends on the remaining “reduced KID equations”. As for a light-cone with vertex at past timelike infinity it turns out that the KID equations adopt at infinity a significantly simpler form as compared to “ordinary” Cauchy surfaces (cf. Theorem 4.1).

2 Setting

2.1 Conformal field equations

In 3+1 dimensions Friedrich’s *metric conformal field equations (MCFE)* (cf. [5])¹

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0 , \quad (2.1)$$

$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta d_{\nu\mu\sigma}{}^\rho , \quad (2.2)$$

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} , \quad (2.3)$$

$$\nabla_\mu s = -L_{\mu\nu} \nabla^\nu \Theta , \quad (2.4)$$

$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3 , \quad (2.5)$$

$$R_{\mu\nu\sigma}{}^\kappa [g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu} L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}) \quad (2.6)$$

form a closed system of equations for the unknowns $g_{\mu\nu}$, Θ , s , $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$. The tensor field $L_{\mu\nu}$ denotes the Schouten tensor,

$$L_{\mu\nu} = \frac{1}{2}R_{\mu\nu} - \frac{1}{12}Rg_{\mu\nu} , \quad (2.7)$$

while

$$d_{\mu\nu\sigma}{}^\rho = \Theta^{-1}C_{\mu\nu\sigma}{}^\rho \quad (2.8)$$

is a rescaling of the conformal Weyl tensor $C_{\mu\nu\sigma}{}^\rho$. The function s is defined as

$$s = \frac{1}{4}\square_g \Theta + \frac{1}{24}R\Theta . \quad (2.9)$$

Friedrich has shown that the MCFE are equivalent to Einstein’s vacuum field equations with cosmological constant λ in regions where the conformal factor Θ , relating the “unphysical” metric $g = \Theta^2 g_{\text{phys}}$ with the physical metric g_{phys} , is positive. Their advantage lies in the property that they remain regular even where Θ vanishes.

The system (2.1)-(2.6) treats s , $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ as independent of $g_{\mu\nu}$ and Θ . However, once a solution of the MCFE has been given these fields are related to $g_{\mu\nu}$ and Θ via (2.7)-(2.9). A solution of the MCFE is thus completely determined by the pair $(g_{\mu\nu}, \Theta)$.

¹It is indicated in [10] that things are considerably different in higher dimensions, which is why we restrict attention to 4 dimensions from the outset.

2.2 Gauge freedom

2.2.1 Conformal factor

Let $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ be some smooth solution of the MCFE.² From $g_{\mu\nu}$ we compute R . Let us then conformally rescale the metric, $g \mapsto \phi^2 g$, for some positive function $\phi > 0$. The Ricci scalars R and R^* of g and $\phi^2 g$, respectively, are related via (set $\square_g := g^{\mu\nu} \nabla_\mu \nabla_\nu$)

$$\phi R - \phi^3 R^* = 6 \square_g \phi. \quad (2.10)$$

Now, let us *prescribe* R^* and read (2.10) as an equation for ϕ . When dealing with a Cauchy problem with data on some spacelike hypersurface \mathcal{H} (including \mathcal{I}^- for $\lambda > 0$) we are free to prescribe functions $\phi|_{\mathcal{H}} =: \mathring{\phi} > 0$ and $\partial_0 \phi|_{\mathcal{H}} =: \mathring{\psi}$ on \mathcal{H} .³ Throughout $x^0 \equiv t$ denotes a time-coordinate so that ∂_0 is transverse to \mathcal{H} . According to standard results there exists a unique solution $\phi > 0$ in some neighborhood of \mathcal{H} which induces the above data on \mathcal{H} . The MCFE are conformally covariant, meaning that the conformally rescaled fields

$$g^* = \phi^2 g, \quad (2.11)$$

$$\Theta^* = \phi \Theta, \quad (2.12)$$

$$s^* = \frac{1}{4} \square_{g^*} \Theta^* + \frac{1}{24} R^* \Theta^*, \quad (2.13)$$

$$L_{\mu\nu}^* = \frac{1}{2} R_{\mu\nu}^*[g^*] - \frac{1}{12} R^* g_{\mu\nu}^*, \quad (2.14)$$

$$d_{\mu\nu\sigma}^{*\rho} = \phi^{-1} d_{\mu\nu\sigma}{}^\rho, \quad (2.15)$$

provide another solution of the MCFE, now with Ricci scalar R^* , which represents the same physical solution: If the conformal factor Θ is treated as an unknown, determined by the MCFE, the unphysical Ricci scalar R can be arranged to adopt any preassigned form, it represents a *conformal gauge source function*.

There remains the gauge freedom to prescribe the functions $\mathring{\phi}$ and $\mathring{\psi}$ on \mathcal{H} . On an ordinary hypersurface, where Θ has no zeros, this freedom can be used to prescribe $\Theta|_{\mathcal{H}}$ and $\partial_0 \Theta|_{\mathcal{H}}$. A main object of this work is to treat the case $\mathcal{H} = \mathcal{I}^-$, where, by definition, $\Theta = 0$ (and $d\Theta \neq 0$). We shall show that in this situation the gauge freedom allows one to prescribe the function s on \mathcal{I}^- and to make conformal rescalings of the induced metric on \mathcal{I}^- .

To see this we consider a smooth solution of the MCFE to the future of \mathcal{I}^- . Now (2.5) and $d\Theta|_{\mathcal{I}^-} \neq 0$ enforce $\mathring{\gamma}^{00} < 0$ (hence, as is well known, \mathcal{I}^- must be spacelike when $\lambda > 0$). Due to (2.5), the function s can be written away from \mathcal{I}^- as

$$s = \frac{1}{2} \Theta^{-1} \nabla_\mu \Theta \nabla^\mu \Theta + \frac{1}{6} \Theta^{-1} \lambda,$$

and the right-hand side is smoothly extendable at \mathcal{I}^- . A conformal rescaling

$$\Theta \mapsto \Theta^* := \phi \Theta, \quad g_{\mu\nu} \mapsto g_{\mu\nu}^* := \phi^2 g_{\mu\nu}, \quad \phi > 0, \quad (2.16)$$

²For convenience we restrict attention throughout to the smooth case, though similar results can be obtained assuming finite differentiability.

³The positivity-assumption on $\mathring{\phi}$ makes sure that the solution of (2.10) is positive sufficiently close to \mathcal{H} and thereby that the new conformal factor Θ^* is positive as well (in the \mathcal{I}^- -case just off the initial surface).

maps the function s to

$$s^* = \phi^{-1} \left(\frac{1}{2} \Theta \phi^{-2} \nabla^\mu \phi \nabla_\mu \phi + \phi^{-1} \nabla^\mu \Theta \nabla_\mu \phi + s \right). \quad (2.17)$$

The trace of this equation on \mathcal{I}^- is

$$\overline{\nabla^\mu \Theta \nabla_\mu \phi + \phi s - \phi^2 s^*} = 0, \quad (2.18)$$

or, in coordinates adapted to \mathcal{I}^- , i.e. for which $\mathcal{I}^- = \{x^0 \equiv t = 0\}$ locally,

$$\overline{g^{0\mu} \nabla_0 \Theta \nabla_\mu \phi + \phi s - \phi^2 s^*} = 0. \quad (2.19)$$

Here and henceforth we use overlining to denote restriction to the initial surface. Let us prescribe $\overline{s^*}$. We choose any $\mathring{\phi} > 0$ to conformally rescale the induced metric on \mathcal{I}^- . Then we solve (2.19) for $\mathring{\psi} \equiv \overline{\nabla_0 \phi}$ (recall that $\overline{\nabla_0 \Theta}$ and \overline{g}^{00} are not allowed to have zeros on \mathcal{I}^-). We take the so-obtained functions $\mathring{\phi} > 0$ and $\mathring{\psi}$ as initial data for (2.10).

By way of summary, the conformal covariance of the MCFE comprises a gauge freedom due to which the functions R and $s|_{\mathcal{I}^-}$ can be regarded as gauge source functions, and due to which only the conformal class of the induced metric on \mathcal{I}^- matters.

2.2.2 Coordinates

It is well-known (cf. e.g. [2]) that the freedom to choose coordinates near a spacelike hypersurface $\mathcal{H} = \{x^0 = 0\}$ with induced Riemannian metric h_{ij} can be employed to prescribe

$$\overline{g}^{00} < 0 \quad \text{and} \quad \overline{g}^{0i}. \quad (2.20)$$

Equivalently, one may prescribe

$$\overline{g}_{00} \quad \text{and} \quad \overline{g}_{0i} \quad \text{such that} \quad \overline{g}_{00} - \overline{h}^{ij} \overline{g}_{0i} \overline{g}_{0j} < 0. \quad (2.21)$$

The remaining freedom to choose coordinates off the initial surface is comprised in the *\hat{g} -generalized wave-map gauge condition*

$$H^\sigma = 0 \quad (2.22)$$

with

$$H^\sigma := g^{\alpha\beta} (\Gamma_{\alpha\beta}^\sigma - \hat{\Gamma}_{\alpha\beta}^\sigma) - W^\sigma \quad (2.23)$$

being the *generalized wave-gauge vector*. Here $\hat{g}_{\mu\nu}$ denotes some *target metric*, $\hat{\Gamma}_{\alpha\beta}^\sigma$ are the Christoffel symbols of $\hat{g}_{\mu\nu}$. More precisely, the gauge freedom is captured by the vector field

$$W^\sigma = W^\sigma(x^\mu, g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \hat{g}_{\mu\nu})$$

which can be arbitrarily prescribed. In fact, within our setup, it can be allowed to depend upon the coordinates, and possibly upon $g_{\mu\nu}$ as well as all other fields which appear in the MCFE, but not upon derivatives thereof.

2.2.3 Realization of the gauge scheme

Given some smooth solution of the MCFE and a new choice of gauge functions R , \bar{s} , W^σ , $\bar{g}_{0\mu}$, as well as a conformal factor $\Omega > 0$ by which one wants to rescale the induced metric \bar{g}_{ij} , a transformation into the new gauge is realized as follows:

In the first step we set $\phi := \Omega$ and solve (2.19) for $\psi \equiv \overline{\nabla_0 \phi}$, which gives us the relevant initial data for (2.10) which we then solve. This way \bar{s} and R take their desired values, and a new representative $\Omega^2 \bar{g}_{ij}$ of the conformal class of the induced metric on \mathcal{S}^- is selected. Then the coordinates are transformed in such a way that the metric takes the prescribed values for $\bar{g}_{0\mu}$ on \mathcal{S}^- . Finally we just need to solve another wave equation to obtain $H^\sigma = 0$ for the given vector field W^σ .

2.3 Constraint equations in the $(R = 0, W^\lambda = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu})$ -wave map gauge

In the following we aim to derive the constraint equations for the fields $g_{\mu\nu}$, Θ , s , $L_{\mu\nu}$, $d_{\mu\nu\sigma}{}^\rho$ on \mathcal{S}^- as well as their transverse derivatives induced by the MCFE on a spacelike \mathcal{S}^- in adapted coordinates $(x^0 = t, x^i)$ with $\mathcal{S}^- = \{t = 0\}$. The surface \mathcal{S}^- is characterized by

$$\bar{\Theta} = 0 \quad \text{and} \quad \overline{d\Theta} \neq 0. \quad (2.24)$$

Note that for \mathcal{S}^- to be spacelike a positive cosmological constant $\lambda > 0$ is required. The constraint equations will be relevant for the derivation of the KID equations in Section 4.

To simplify computations we make the specific gauge choice

$$R = 0, \quad \bar{s} = 0, \quad \bar{g}_{00} = -1, \quad \bar{g}_{0i} = 0, \quad W^\sigma = 0, \quad \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu}. \quad (2.25)$$

(Note that the target metric is taken to be $\bar{g}_{\mu\nu}$ for all t .) We shall show that appropriate data to solve the constraint equations are \bar{g}_{ij} and \bar{d}_{0i0j} , where the latter field needs to satisfy a vector and a scalar constraint equation.

Let us start with a list of all the Christoffel symbols in adapted coordinates

$$\bar{\Gamma}_{ij}^k = \tilde{\Gamma}_{ij}^k, \quad \bar{\Gamma}_{ij}^0 = \frac{1}{2} \overline{\partial_0 g_{ij}}, \quad \bar{\Gamma}_{0i}^0 = 0, \quad (2.26)$$

$$\bar{\Gamma}_{00}^0 = -\frac{1}{2} \overline{\partial_0 g_{00}}, \quad \bar{\Gamma}_{00}^k = \bar{g}^{kl} \overline{\partial_0 g_{0l}}, \quad \bar{\Gamma}_{0i}^k = \frac{1}{2} \bar{g}^{kl} \overline{\partial_0 g_{il}}, \quad (2.27)$$

where the $\tilde{\Gamma}_{ij}^k$'s denote the Christoffel symbols of the Riemannian metric $\tilde{g} = \bar{g}_{ij} dx^i dx^j$. Throughout we shall use $\tilde{\cdot}$ to denote fields such as the Riemann tensor, the Levi-Civita connection etc. associated to \tilde{g} .

Evaluation of (2.5) on \mathcal{S}^- gives

$$\overline{\nabla_0 \Theta} = \sqrt{\lambda/3}. \quad (2.28)$$

The $(\mu\nu) = (00)$ -component of (2.3) implies

$$\overline{\nabla_0 \nabla_0 \Theta} = 0, \quad (2.29)$$

while the $(\mu\nu) = (ij)$ -components of (2.3) yield

$$0 = \overline{\nabla_i \nabla_j \Theta} = -\bar{\Gamma}_{ij}^0 \overline{\nabla_0 \Theta} = -\sqrt{\frac{\lambda}{12}} \overline{\partial_0 g_{ij}}. \quad (2.30)$$

We compute the $(\mu\nu\sigma\kappa) = (ikjk)$ -components of (2.6),

$$\overline{R}_{ikj}{}^k = \overline{L}_{ij} + g_{ij}\overline{g}^{kl}\overline{L}_{kl} ,$$

where

$$\overline{R}_{ikj}{}^k = \partial_k \overline{\Gamma}_{ij}^k - \partial_i \overline{\Gamma}_{jk}^k + \overline{\Gamma}_{ij}^\alpha \overline{\Gamma}_{\alpha k}^k - \overline{\Gamma}_{ik}^\alpha \overline{\Gamma}_{j\alpha}^k = \tilde{R}_{ikj}{}^k = \tilde{R}_{ij} .$$

Hence

$$\overline{L}_{ij} = \tilde{R}_{ij} - \frac{1}{4}\overline{g}_{ij}\tilde{R} = \tilde{L}_{ij} , \quad (2.31)$$

where \tilde{L}_{ij} is the *Schouten tensor* of \tilde{g} . The gauge conditions (2.25) imply

$$0 = \frac{1}{6}\overline{R} = \overline{g}^{\mu\nu}\overline{L}_{\mu\nu} = \overline{g}^{ij}\overline{L}_{ij} - \overline{L}_{00} = \frac{1}{4}\tilde{R} - \overline{L}_{00} . \quad (2.32)$$

From the $\mu = i$ -component of (2.4) we deduce

$$\overline{L}_{0i} = 0 . \quad (2.33)$$

Next, we employ the wave-map gauge condition to obtain

$$\begin{aligned} 0 &= \overline{H}^k = g^{\alpha\beta}(\Gamma_{\alpha\beta}^k - \hat{\Gamma}_{\alpha\beta}^k) = -\overline{\Gamma}_{00}^k = -\overline{g}^{kl}\overline{\partial_0 g_{0l}} , \\ 0 &= \overline{H}^0 = g^{\alpha\beta}(\Gamma_{\alpha\beta}^0 - \hat{\Gamma}_{\alpha\beta}^0) = -\overline{\Gamma}_{00}^0 = \frac{1}{2}\overline{\partial_0 g_{00}} . \end{aligned}$$

Altogether we have found that

$$\overline{\partial_0 g_{\mu\nu}} = 0 . \quad (2.34)$$

Thus (2.26)-(2.27) simplify to

$$\overline{\Gamma}_{ij}^k = \tilde{\Gamma}_{ij}^k , \quad \overline{\Gamma}_{ij}^0 = \overline{\Gamma}_{0i}^0 = \overline{\Gamma}_{00}^0 = \overline{\Gamma}_{00}^k = \overline{\Gamma}_{0i}^k = 0 . \quad (2.35)$$

We have

$$\begin{aligned} \overline{R}_{ij} &\equiv \overline{\partial_\mu \Gamma_{ij}^\mu} - \overline{\partial_i \Gamma_{j\mu}^\mu} + \overline{\Gamma_{ij}^\alpha \Gamma_{\alpha\mu}^\mu} - \overline{\Gamma_{i\mu}^\alpha \Gamma_{j\alpha}^\mu} \\ &= \tilde{R}_{ij} + \overline{\partial_0 \Gamma_{ij}^0} = \tilde{R}_{ij} + \frac{1}{2}\overline{\partial_0 \partial_0 g_{ij}} . \end{aligned}$$

Hence

$$\overline{\partial_0 \partial_0 g_{ij}} = 4\overline{L}_{ij} - 2\tilde{R}_{ij} = 2\tilde{R}_{ij} - g_{ij}\tilde{R} . \quad (2.36)$$

If we evaluate the $\mu = 0$ -component of (2.4) on \mathcal{S}^- we are led to,

$$\overline{\nabla_0 s} = \overline{L}_{00}\overline{\nabla_0 \Theta} = \sqrt{\frac{\lambda}{48}}\tilde{R} . \quad (2.37)$$

The $(\mu\nu\sigma) = (0i0)$ -components of (2.2) yield

$$\overline{\nabla_0 L_{0i}} = \nabla_i \overline{L}_{00} = \frac{1}{4}\tilde{\nabla}_i \tilde{R} . \quad (2.38)$$

Moreover, for $(\mu\nu\sigma) = (jki)$ we obtain

$$\bar{d}_{0ijk} = \sqrt{\frac{12}{\lambda}} \tilde{\nabla}_{[k} \bar{L}_{j]i} = \sqrt{\frac{3}{\lambda}} \tilde{C}_{ijk} , \quad (2.39)$$

where \tilde{C}_{ijk} is the *Cotton tensor* of \tilde{g} . For $(\mu\nu\sigma) = (0ji)$ we find

$$\overline{\nabla_0 L_{ij}} = -\sqrt{\lambda/3} \bar{d}_{0i0j} . \quad (2.40)$$

The gauge condition $R = 0$ together with the tracelessness of the rescaled Weyl tensor then imply

$$0 = \bar{g}^{\mu\nu} \overline{\nabla_0 L_{\mu\nu}} = \bar{g}^{ij} \overline{\nabla_0 L_{ij}} - \overline{\nabla_0 L_{00}} = -\overline{\nabla_0 L_{00}} . \quad (2.41)$$

Via the second Bianchi identity the $(\mu\nu\sigma) = (0ij)$ -components of (2.1) become

$$\overline{\nabla_0 d_{0i0j}} = -\tilde{\nabla}^k \bar{d}_{0ijk} = -\sqrt{\frac{3}{\lambda}} \tilde{\nabla}^k \tilde{C}_{ijk} = \sqrt{\frac{3}{\lambda}} \tilde{B}_{ij} , \quad (2.42)$$

where \tilde{B}_{ij} denotes the *Bach tensor* of \tilde{g} . The $(\mu\nu\sigma) = (kji)$ -components give

$$\overline{\nabla_0 d_{0ijk}} = -\tilde{\nabla}^l \bar{d}_{jkil} = 2\tilde{\nabla}_{[j} \bar{d}_{k]0i0} - 2g_{i[j} \tilde{\nabla}^l \bar{d}_{k]0l0} . \quad (2.43)$$

Here we used that due to the algebraic symmetries of the rescaled Weyl tensor

$$\begin{aligned} \bar{d}_{ijkl} &= 2\bar{g}^{mn} (\bar{g}_{k[i} \bar{d}_{j]mln} - \bar{g}_{l[i} \bar{d}_{j]mkn} - \bar{g}_{k[i} g_{j]l} \bar{g}^{pq} \bar{d}_{pmqn}) \\ &= 2(\bar{g}_{k[i} \bar{d}_{j]0l0} - \bar{g}_{l[i} \bar{d}_{j]0k0}) . \end{aligned} \quad (2.44)$$

The $(\mu\nu\sigma) = (0i0)$ -components of (2.1) imply a *vector constraint* for \bar{d}_{0i0j} ,

$$\tilde{\nabla}^j \bar{d}_{0i0j} = 0 . \quad (2.45)$$

(A “*scalar constraint*”, which has already been used in the derivation of the constraint equations, is simply given by the tracelessness-requirement on the rescaled Weyl tensor,

$$\bar{g}^{ij} \bar{d}_{0i0j} = \bar{g}^{\mu\nu} \bar{d}_{0\mu 0\nu} = 0 .) \quad (2.46)$$

To sum it up, we have the following analogue of a result of Friedrich [4]: The free data can be identified with a Riemannian metric $h_{ij} := \bar{g}_{ij}$ and a symmetric tensor field $D_{ij} := \bar{d}_{0i0j}$ on \mathcal{S}^- satisfying

$$h^{ij} D_{ij} = 0 \quad \text{and} \quad \tilde{\nabla}^j D_{ij} = 0 \quad (2.47)$$

(that these are indeed the free data follows e.g. from the considerations in Appendix A). Then the MCFE enforce on \mathcal{S}^- in the $(R = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu})$ -wave-map gauge,

$$\bar{g}_{00} = -1 , \quad \bar{g}_{0i} = 0 , \quad \bar{g}_{ij} = h_{ij} , \quad \bar{\partial}_0 g_{\mu\nu} = 0 , \quad (2.48)$$

$$\bar{\Theta} = 0 , \quad \bar{\partial}_0 \bar{\Theta} = \sqrt{\frac{\lambda}{3}} , \quad (2.49)$$

$$\bar{s} = 0 , \quad \bar{\partial}_0 \bar{s} = \sqrt{\frac{\lambda}{48}} \tilde{R} , \quad (2.50)$$

$$\bar{L}_{ij} = \tilde{L}_{ij} , \quad \bar{L}_{0i} = 0 , \quad \bar{L}_{00} = \frac{1}{4} \tilde{R} , \quad (2.51)$$

$$\overline{\partial_0 L_{ij}} = -\sqrt{\frac{\lambda}{3}} D_{ij} , \quad \overline{\partial_0 L_{0i}} = \frac{1}{4} \tilde{\nabla}_i \tilde{R} , \quad \overline{\partial_0 L_{00}} = 0 , \quad (2.52)$$

$$\bar{d}_{0i0j} = D_{ij} , \quad \bar{d}_{0ijk} = \sqrt{\frac{3}{\lambda}} \tilde{C}_{ijk} , \quad (2.53)$$

$$\overline{\partial_0 d_{0i0j}} = \sqrt{\frac{3}{\lambda}} \tilde{B}_{ij} , \quad \overline{\partial_0 d_{0ijk}} = 2\tilde{\nabla}_{[j} D_{k]i} . \quad (2.54)$$

Note that due to (2.35) the actions of ∇_0 and ∂_0 , as well as ∇_i and $\tilde{\nabla}_i$, respectively, coincide on \mathcal{I}^- , so we can use them interchangeably.

We have seen in Section 2.2 (cf. also [4]) that there remains a gauge freedom to conformally rescale the induced metric on \mathcal{I}^- . Due to this freedom the pairs (h_{ij}, D_{ij}) and $(\Omega^2 h_{ij}, \Omega^{-1} D_{ij})$, with Ω some positive function, generate the same physical space-times. With regard to the constraint equations we note that $\Omega^{-1} D_{ij}$ is trace- and divergence-free w.r.t. $\Omega^2 h_{ij}$ whenever D_{ij} is w.r.t. h_{ij} .

In the following we shall write $[h_{ij}, D_{ij}]$ if this gauge freedom is left unspecified and if we merely want to refer to the conformal classes of h_{ij} and D_{ij} .

2.4 Well-posedness of the Cauchy problem on a space-like \mathcal{I}^-

In [9] a system of conformal wave equations (CWE) has been derived from the MCFE. In Appendix A it is shown that a solution of the CWE, equations (A.1)-(A.5), is a solution of the MCFE if and only if the constraint equations (2.47)-(2.54) are satisfied. Using standard well-posedness results about wave equations we thereby recover a result due to Friedrich [4] who proved well-posedness of the Cauchy problem on \mathcal{I}^- (Friedrich used a representation of the MCFE as a symmetric hyperbolic system, in some situations, however, it might be advantageous to deal with a system of wave equations instead [6]). We restrict attention to the smooth case (for a version with finite differentiability see [4]):

THEOREM 2.1 *Let \mathcal{H} be a 3-dimensional smooth manifold. Let h_{ij} be a smooth Riemannian metric and let D_{ij} be a smooth symmetric, trace- and divergence-free tensor field on \mathcal{H} . Moreover, assume a positive cosmological constant $\lambda > 0$. Then there exists an (up to isometries) unique smooth space-time (\mathcal{M}, g, Θ) with the following properties:*

- (i) (\mathcal{M}, g, Θ) satisfies the MCFE (2.1)-(2.6),
- (ii) $\Theta|_{\mathcal{H}} = 0$ and $d\Theta|_{\mathcal{H}} \neq 0$, i.e. $\mathcal{H} = \mathcal{I}^-$ (and Θ has no zeros away from and sufficiently close to \mathcal{H}),
- (iii) $g_{ij}|_{\mathcal{H}} = h_{ij}$, $d_{0i0j}|_{\mathcal{H}} = D_{ij}$.

The isometry class of the space-time does not change if the initial data are replaced by $(\hat{h}_{ij}, \hat{D}_{ij})$ with $[\hat{h}_{ij}, \hat{D}_{ij}] = [h_{ij}, D_{ij}]$.

REMARK 2.2 De Sitter space-time is obtained for $\mathcal{H} = S^3$, $h_{ij} = s_{ij}$ and $D_{ij} = 0$, where $s = s_{ij} dx^i dx^j$ denotes the round sphere metric, cf. Section 4.3.2

3 KID equations

3.1 Unphysical Killing equations

In [10] it is shown that the appropriate substitute for the Killing equation in the unphysical, conformally rescaled space-time is provided by the *unphysical Killing equations*

$$\nabla_{(\mu} X_{\nu)} = \frac{1}{4} \nabla^\sigma X_\sigma g_{\mu\nu} \quad \& \quad X^\sigma \nabla_\sigma \Theta = \frac{1}{4} \Theta \nabla_\sigma X^\sigma. \quad (3.1)$$

A vector field X_{phys} is a Killing field in the physical space-time $(\mathcal{M}_{\text{phys}}, g_{\text{phys}})$ if and only if its push-forward $X := \phi_* X_{\text{phys}}$ satisfies (3.1) in the unphysical space-time $(\phi(\mathcal{M}_{\text{phys}}) \subset \mathcal{M}, g = \phi(g_{\text{phys}}) = \Theta^2 g_{\text{phys}})$, where ϕ defines the conformal rescaling. The unphysical Killing equations remain regular even where the conformal factor Θ vanishes.

In what follows we shall derive necessary-and-sufficient conditions on a space-like initial surface which guarantee the existence of a vector field X which satisfies the unphysical Killing equations.

3.2 KID equations on a Cauchy surface

Necessary conditions on a vector field X to satisfy the unphysical Killing equations are that the following wave equations are fulfilled [10],

$$\square_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y = 0, \quad (3.2)$$

$$\square_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} RY = 0, \quad (3.3)$$

where we have set

$$Y := \frac{1}{4} \nabla_\sigma X^\sigma. \quad (3.4)$$

It proves fruitful to make the following definitions:

$$\phi := X^\mu \nabla_\mu \Theta - \Theta Y, \quad (3.5)$$

$$\psi := X^\mu \nabla_\mu s + sY - \nabla_\mu \Theta \nabla^\mu Y, \quad (3.6)$$

$$A_{\mu\nu} := 2\nabla_{(\mu} X_{\nu)} - 2Y g_{\mu\nu}, \quad (3.7)$$

$$B_{\mu\nu} := \mathcal{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y. \quad (3.8)$$

All these fields need to vanish whenever X is a solution of (3.1) [10].

The equations (3.2) and (3.3) together with the MCFE imply that the following system of wave equations is satisfied by the fields ϕ , ψ , $A_{\mu\nu}$, $\nabla_\sigma A_{\mu\nu}$ and $B_{\mu\nu}$ (cf. [10]):

$$\square_g A_{\mu\nu} = 2R_{(\mu}{}^\kappa A_{\nu)\kappa} - 2R_\mu{}^\alpha{}_\nu{}^\beta A_{\alpha\beta} - 4B_{\mu\nu}, \quad (3.9)$$

$$\square_g \phi = d\psi - \frac{1}{6} R\phi + A_{\mu\nu} \nabla^\mu \nabla^\nu \Theta, \quad (3.10)$$

$$\begin{aligned} \square_g \psi &= |L|^2 \phi + A_{\mu\nu} (\nabla^\mu \nabla^\nu s - 2\Theta L_\kappa{}^\mu L^{\nu\kappa}) + 2\Theta L^{\mu\nu} B_{\mu\nu} \\ &\quad + \frac{1}{6} (A_{\mu\nu} \nabla^\mu R \nabla^\nu \Theta - \nabla^\mu R \nabla_\mu \phi - R\psi), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \square_g B_{\mu\nu} &\equiv 2(g_{\mu\nu} L^{\alpha\beta} - R_\mu{}^\alpha{}_\nu{}^\beta) B_{\alpha\beta} - 2R_{(\mu}{}^\kappa B_{\nu)\kappa} + \frac{2}{3} R B_{\mu\nu} \\ &\quad + 2L^{\alpha\beta} (\nabla_\beta \nabla_{[\alpha} A_{\nu]\mu} - \nabla_\mu \nabla_{[\alpha} A_{\nu]\beta}) \\ &\quad + (\nabla_{(\mu} A_{|\alpha\beta|} + 2\nabla_{[\alpha} A_{\beta](\mu}) (2\nabla^\alpha L_{\nu)}{}^\beta - \frac{1}{12} \delta_{\nu)}{}^\alpha \nabla^\beta R) \\ &\quad + A^{\alpha\beta} [\nabla_\alpha \nabla_\beta L_{\mu\nu} - 2L_{(\mu}{}^\kappa R_{\nu)\alpha\kappa\beta} + 2L_{\mu\alpha} R_{\nu\beta} + L_\alpha{}^\kappa (2R_{\mu\beta\nu\kappa} + R_{\nu\beta\mu\kappa}) \\ &\quad - 2g_{\mu\nu} L_{\alpha\kappa} L_\beta{}^\kappa] + |L|^2 A_{\mu\nu} + L^{\alpha\beta} R_{\mu\alpha\beta}{}^\kappa A_{\nu\kappa} - \frac{1}{3} R L_{(\mu}{}^\kappa A_{\nu)\kappa}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \square_g \nabla_\sigma A_{\mu\nu} &= 2\nabla_\sigma (R_{(\mu}{}^\kappa A_{\nu)\kappa} - R_\mu{}^\alpha{}_\nu{}^\kappa A_{\alpha\kappa}) + 2A_{\alpha(\mu} (\nabla_{\nu)} R_\sigma{}^\alpha - \nabla^\alpha R_{\nu)\sigma}) \\ &\quad - 4R_{\sigma\kappa(\mu}{}^\alpha \nabla^\kappa A_{\nu)\alpha} + R_{\alpha\sigma} \nabla^\alpha A_{\mu\nu} - 4\nabla_\sigma B_{\mu\nu}. \end{aligned} \quad (3.13)$$

In close analogy to [10, Theorem 3.4] we immediately obtain the following result:

THEOREM 3.1 *Assume we have been given, in 3+1 dimensions, an “unphysical” space-time (\mathcal{M}, g, Θ) , with (g, Θ) a smooth solution of the MCFE (2.1)-(2.6). Consider a spacelike hypersurface $\mathcal{H} \subset \mathcal{M}$. Then there exists a vector field \hat{X} satisfying the unphysical Killing equations (3.1) on $D^+(\mathcal{H})$ (and thus corresponding to a Killing vector field of the physical space-time) if and only if there exists a pair (X, Y) , X a vector field and Y a function, which fulfills the following equations:*

$$(i) \quad \square_g X_\mu + R_\mu{}^\nu X_\nu + 2\nabla_\mu Y = 0,$$

$$(ii) \quad \square_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} RY = 0,$$

$$(iii) \quad \bar{\phi} = 0 \text{ and } \bar{\partial}_0 \bar{\phi} = 0,$$

$$(iv) \quad \bar{\psi} = 0 \text{ and } \bar{\partial}_0 \bar{\psi} = 0,$$

$$(v) \quad \bar{A}_{\mu\nu} = 0, \quad \bar{\nabla}_0 \bar{A}_{\mu\nu} = 0 \text{ and } \bar{\nabla}_0 \bar{\nabla}_0 \bar{A}_{\mu\nu} = 0,$$

$$(vi) \quad \bar{B}_{\mu\nu} = 0 \text{ and } \bar{\nabla}_0 \bar{B}_{\mu\nu} = 0.$$

Moreover, $\bar{\hat{X}} = \bar{X}$, $\bar{\nabla}_0 \bar{\hat{X}} = \bar{\nabla}_0 \bar{X}$, $\bar{\nabla}_\mu \bar{\hat{X}}^\mu = \frac{1}{4} \bar{Y}$ and $\bar{\nabla}_0 \bar{\nabla}_\mu \bar{\hat{X}}^\mu = \frac{1}{4} \bar{\nabla}_0 \bar{Y}$.

3.3 A special case: $\Theta = 1$

Let us briefly discuss the case where the conformal factor Θ is identical to one,

$$\Theta = 1,$$

so that the unphysical space-time can be identified with the physical one. Then the MCFE imply

$$s = \frac{1}{6} \lambda, \quad L_{\mu\nu} = s g_{\mu\nu}, \quad R_{\mu\nu} = \lambda g_{\mu\nu},$$

i.e. the vacuum Einstein equations hold. We consider the conditions (i)-(vi) of Theorem 3.1 in this setting. Condition (iii) is equivalent to $\bar{Y} = 0$ and $\bar{\partial}_0 \bar{Y} = 0$, which provide the initial data for the wave equation (ii). The only solution is $Y = 0$, i.e. X needs to be a Killing field, as desired. Condition (iv) is then automatically satisfied. Since

$$B_{\mu\nu} = \mathcal{L}_X L_{\mu\nu} = s \mathcal{L}_X g_{\mu\nu} = 2s \nabla_{(\mu} X_{\nu)}, \quad (3.14)$$

the validity of (vi) follows from (v), and we are left with the conditions

$$\square_g X_\mu + \lambda X_\mu = 0, \quad (3.15)$$

$$\bar{\nabla}_{(\mu} X_{\nu)} = 0, \quad (3.16)$$

$$\bar{\nabla}_0 \bar{\nabla}_{(\mu} X_{\nu)} = 0, \quad (3.17)$$

$$\bar{\nabla}_0 \bar{\nabla}_0 \bar{\nabla}_{(\mu} X_{\nu)} = 0. \quad (3.18)$$

Note that $\bar{B}_{\mu\nu} = 0$ due to (3.14) and (3.16), so that (3.15)-(3.17) imply via the trace of (3.9) on \mathcal{H} the validity of (3.18).

The equations (3.15)-(3.17) form a possible starting point to derive the KID equations on Cauchy surfaces in space-times satisfying the vacuum Einstein equations (cf. [1, 8]).

3.4 A stronger version of Theorem 3.1

Let us now investigate to what extent the conditions (iii)-(vi) in Theorem 3.1 imply each other. For this purpose we choose adapted coordinates $(x^0 \equiv t, x^i)$ in the sense that the initial surface is (locally) given by the set $\{x^0 = 0\}$ and that, on \mathcal{H} , the metric takes the form

$$g|_{\mathcal{H}} = -(\mathrm{d}t)^2 + \bar{g}_{ij} \mathrm{d}x^i \mathrm{d}x^j = -(\mathrm{d}t)^2 + h_{ij} \mathrm{d}x^i \mathrm{d}x^j, \quad (3.19)$$

with h_{ij} some Riemannian metric. Moreover, we denote by f , f_i and f_{ij} generic functions which depend on the indicated fields (and possibly spatial derivatives thereof) and vanish whenever all their arguments vanish. The symbol $\check{\cdot}$ is used to denote the h -trace-free part of the corresponding 2-rank tensor on \mathcal{H} , i.e.

$$\check{v}_{ij} := v_{ij} - \frac{1}{3} h_{ij} h^{kl} v_{kl}. \quad (3.20)$$

We start with the identity [10]

$$\nabla_\nu A_\mu{}^\nu - \frac{1}{2} \nabla_\mu A_\nu{}^\nu \equiv \square_g X_\mu + R_\mu{}^\nu X_\nu + 2 \nabla_\mu Y. \quad (3.21)$$

Because of (3.2) the right-hand side vanishes and we obtain

$$\overline{\nabla_0 A_{00}} = 2\bar{g}^{kl} \nabla_k \bar{A}_{0l} - \bar{g}^{kl} \overline{\nabla_0 A_{kl}} = -\bar{g}^{kl} \overline{\nabla_0 A_{kl}} + f(\bar{A}_{\mu\nu}), \quad (3.22)$$

$$\overline{\nabla_0 A_{0i}} = \frac{1}{2} \nabla_i \bar{A}_{00} + \bar{g}^{kl} \nabla_k \bar{A}_{il} - \frac{1}{2} \bar{g}^{kl} \nabla_i \bar{A}_{kl} = f_i(\bar{A}_{\mu\nu}), \quad (3.23)$$

$$\begin{aligned} \overline{\nabla_0 \nabla_0 A_{00}} &= 2\bar{g}^{kl} \overline{\nabla_0 \nabla_k A_{0l}} - \bar{g}^{kl} \overline{\nabla_0 \nabla_0 A_{kl}} \\ &= 2\bar{g}^{kl} \nabla_k \overline{\nabla_0 A_{0l}} - \bar{g}^{kl} \overline{\nabla_0 \nabla_0 A_{kl}} + f(\bar{A}_{\mu\nu}), \end{aligned} \quad (3.24)$$

$$\begin{aligned} \overline{\nabla_0 \nabla_0 A_{0i}} &= \frac{1}{2} \overline{\nabla_0 \nabla_i A_{00}} + \bar{g}^{kl} \overline{\nabla_0 \nabla_k A_{il}} - \frac{1}{2} \bar{g}^{kl} \overline{\nabla_0 \nabla_i A_{kl}} \\ &= \frac{1}{2} \nabla_i \overline{\nabla_0 A_{00}} + \bar{g}^{kl} \nabla_k \overline{\nabla_0 A_{il}} - \frac{1}{2} \bar{g}^{kl} \nabla_i \overline{\nabla_0 A_{kl}} + f_i(\bar{A}_{\mu\nu}) \end{aligned} \quad (3.25)$$

We further have the identity [10]

$$\begin{aligned} \nabla_\nu B_\mu{}^\nu - \frac{1}{2} \nabla_\mu B_\nu{}^\nu &\equiv A_{\alpha\beta} (\nabla^\alpha L_\mu{}^\beta - \frac{1}{2} \nabla_\mu L^{\alpha\beta}) \\ &+ L_\mu{}^\kappa (\square_g X_\kappa + R_\kappa{}^\alpha X_\alpha + 2 \nabla_\kappa Y) + \frac{1}{2} \nabla_\mu (\square_g Y + \frac{1}{6} X^\nu \nabla_\nu R + \frac{1}{3} RY). \end{aligned}$$

With (3.2) and (3.3) we deduce

$$\begin{aligned} \overline{\nabla_0 B_{00}} &= 2\bar{g}^{kl} \nabla_k \bar{B}_{0l} - \bar{g}^{kl} \overline{\nabla_0 B_{kl}} + f(\bar{A}_{\mu\nu}) \\ &= -\bar{g}^{kl} \overline{\nabla_0 B_{kl}} + f(\bar{A}_{\mu\nu}, \bar{B}_{\mu\nu}), \end{aligned} \quad (3.26)$$

$$\begin{aligned} \overline{\nabla_0 B_{0i}} &= \frac{1}{2} \nabla_i \bar{B}_{00} + \bar{g}^{kl} \nabla_k \bar{B}_{il} - \frac{1}{2} \bar{g}^{kl} \nabla_i \bar{B}_{kl} + f_i(\bar{A}_{\mu\nu}) \\ &= f_i(\bar{A}_{\mu\nu}, \bar{B}_{\mu\nu}). \end{aligned} \quad (3.27)$$

Evaluation of (3.9) on the initial surface gives with (2.26)-(2.27)

$$\overline{\nabla_0 \nabla_0 A_{ij}} = 4\bar{B}_{ij} - \bar{g}^{kl} \bar{\Gamma}_{kl}^0 \overline{\nabla_0 A_{ij}} + f_{ij}(\bar{A}_{\mu\nu}), \quad (3.28)$$

$$\overline{\nabla_0 \nabla_0 A_{0i}} = 4\bar{B}_{0i} - \bar{g}^{kl} \bar{\Gamma}_{kl}^0 \overline{\nabla_0 A_{0i}} + f_i(\bar{A}_{\mu\nu}), \quad (3.29)$$

$$\overline{\nabla_0 \nabla_0 A_{00}} = 4\bar{B}_{00} - \bar{g}^{kl} \bar{\Gamma}_{kl}^0 \overline{\nabla_0 A_{00}} + f(\bar{A}_{\mu\nu}). \quad (3.30)$$

From the definition of $B_{\mu\nu}$ we obtain with (3.3) (set $B := g^{\mu\nu} B_{\mu\nu}$)

$$\begin{aligned}\bar{B} &\equiv \bar{L}^{\mu\nu} \bar{A}_{\mu\nu} + \overline{\square_g Y} + \frac{1}{6} \bar{X}^\mu \bar{\nabla}_\mu \bar{R} + \frac{1}{3} \bar{R} \bar{Y} \\ &= \bar{L}^{\mu\nu} \bar{A}_{\mu\nu},\end{aligned}\tag{3.31}$$

$$\begin{aligned}\overline{\nabla_0 B} &\equiv \overline{\nabla_0(L^{\mu\nu} A_{\mu\nu})} + \overline{\nabla_0(\square_g Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} RY)} \\ &= \overline{\nabla_0(L^{\mu\nu} A_{\mu\nu})}.\end{aligned}\tag{3.32}$$

We use the equations (3.22)-(3.32) to establish a stronger version of Theorem 3.1. Let us assume that

$$\bar{A}_{\mu\nu} = 0, \quad \overline{\nabla_0 A_{ij}} = 0, \quad \check{\bar{B}}_{ij} = 0, \quad (\overline{\nabla_0 B_{ij}})^\vee = 0.\tag{3.33}$$

Then by (3.22) and (3.23) we have $\overline{\nabla_0 A_{\mu\nu}} = 0$. From (3.31) and (3.32) we deduce $\bar{B} = \overline{\nabla_0 B} = 0$. The equations (3.24), (3.28) and (3.30) yield the system

$$\begin{aligned}\overline{\nabla_0 \nabla_0 A_{00}} &= -\bar{g}^{ij} \overline{\nabla_0 \nabla_0 A_{ij}}, \\ \bar{g}^{ij} \overline{\nabla_0 \nabla_0 A_{ij}} &= 4\bar{g}^{ij} \bar{B}_{ij} \stackrel{\bar{B}=0}{=} 4\bar{B}_{00}, \\ \overline{\nabla_0 \nabla_0 A_{00}} &= 4\bar{B}_{00},\end{aligned}$$

from which we conclude $\overline{\nabla_0 \nabla_0 A_{00}} = \bar{g}^{ij} \overline{\nabla_0 \nabla_0 A_{ij}} = \bar{B}_{00} = 0$. From (3.25) and the trace-free part of (3.28) we then deduce $\overline{\nabla_0 \nabla_0 A_{\mu\nu}} = 0$, and the equations (3.29) and (3.31) imply $\bar{B}_{\mu\nu} = 0$. Moreover, invoking (3.26) and (3.32) yields

$$\begin{aligned}\overline{\nabla_0 B_{00}} &= -\bar{g}^{ij} \overline{\nabla_0 B_{ij}}, \\ 0 = \overline{\nabla_0 B} &= \bar{g}^{ij} \overline{\nabla_0 B_{ij}} - \overline{\nabla_0 B_{00}},\end{aligned}$$

i.e. $\overline{\nabla_0 B_{00}} = \bar{g}^{ij} \overline{\nabla_0 B_{ij}} = 0$. The equation (3.27) then completes the proof that $\overline{\nabla_0 B_{\mu\nu}} = 0$.

We end up with the result

THEOREM 3.2 *Assume we have been given, in 3+1 dimensions, an “unphysical” space-time (\mathcal{M}, g, Θ) , with (g, Θ) a smooth solution of the MCFE (2.1)-(2.6). Consider a spacelike hypersurface $\mathcal{H} \subset \mathcal{M}$. Then there exists a vector field \hat{X} satisfying the unphysical Killing equations (3.1) on $D^+(\mathcal{H})$ if and only if there exists a pair (X, Y) , X a vector field and Y a function, which fulfills the KID equations, i.e.*

(a) *equations (i)-(iv) of Theorem 3.1,*

(b) *$\bar{A}_{\mu\nu} = 0$ and $\overline{\nabla_0 A_{ij}} = 0$ with $A_{\mu\nu} \equiv 2\nabla_{(\mu} X_{\nu)} - 2Y g_{\mu\nu}$,*

(c) *$\check{\bar{B}}_{ij} = 0$ and $(\overline{\nabla_0 B_{ij}})^\vee = 0$ with $B_{\mu\nu} \equiv \mathcal{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y$.*

Moreover, $\bar{\hat{X}} = \bar{X}$, $\overline{\nabla_0 \hat{X}} = \overline{\nabla_0 X}$, $\overline{\nabla_\mu \hat{X}^\mu} = \frac{1}{4} \bar{Y}$ and $\overline{\nabla_0 \nabla_\mu \hat{X}^\mu} = \frac{1}{4} \overline{\nabla_0 \hat{Y}}$.

3.5 The (proper) KID equations

We want to replace the equations $\bar{\partial_0 \psi} = 0$ and $(\overline{\nabla_0 B_{ij}})^\vee = 0$ appearing in Theorem 3.2 by intrinsic equations on \mathcal{H} in the sense that they involve at most

first-order transverse derivatives of X and Y , which belong to the freely prescribable initial data for the wave equations (3.2) and (3.3). The higher-order derivatives appearing can be eliminated via (3.3) which implies

$$\overline{\nabla_0 \nabla_0 Y} = \overline{g^{kl} \nabla_k \nabla_l Y} + \frac{1}{6} \overline{X^\mu \nabla_\mu R} + \frac{1}{3} \overline{R Y}. \quad (3.34)$$

We are straightforwardly led to

THEOREM 3.3 *Assume that we have been given a 3 + 1-dimensional space-time (\mathcal{M}, g, Θ) , with (g, Θ) being a smooth solution of the MCFE. Let \dot{X} and $\dot{\Lambda}$ be spacetime vector fields, and \dot{Y} and $\dot{\Upsilon}$ be functions defined along a spacelike hypersurface $\mathcal{H} \subset \mathcal{M}$. Then there exists a smooth space-time vector field X with $\overline{X} = \dot{X}$, $\overline{\nabla_0 X} = \dot{\Lambda}$, $\overline{\nabla_\mu X^\mu} = \frac{1}{4} \dot{Y}$ and $\overline{\nabla_0 \nabla_\mu X^\mu} = \frac{1}{4} \dot{\Upsilon}$ satisfying the unphysical Killing equations (3.1) on $D^+(\mathcal{H})$ (and thus corresponding to a Killing field of the physical space-time) if and only if in the adapted coordinates (3.19):*

$$\begin{aligned} (i) \quad & \overline{\phi} \equiv \dot{X}^\mu \overline{\nabla_\mu \Theta} - \overline{\Theta} \dot{Y} = 0, \\ & \frac{\partial_0 \overline{\phi}}{\partial_0 \phi} \equiv \dot{\Lambda}^\mu \overline{\nabla_\mu \Theta} + \dot{X}^\mu \overline{\nabla_\mu \nabla_0 \Theta} - \overline{\Theta} \dot{\Upsilon} - \overline{\nabla_0 \Theta} \dot{Y} = 0, \\ (ii) \quad & \overline{\psi} \equiv \dot{X}^\mu \overline{\nabla_\mu s} + \overline{s} \dot{Y} - \overline{\nabla^i \Theta} \tilde{\nabla}_i \dot{Y} + \overline{\nabla_0 \Theta} \dot{\Upsilon} = 0, \\ & \frac{\partial_0 \overline{\psi}^{\text{intr}}}{\partial_0 \psi^{\text{intr}}} := \dot{\Lambda}^\mu \overline{\nabla_\mu s} + \dot{X}^\mu \overline{\nabla_\mu \nabla_0 s} + \overline{\nabla_0 s} \dot{Y} + (\overline{s} + \overline{\nabla_0 \nabla_0 \Theta}) \dot{\Upsilon} - \overline{\nabla^i \nabla_0 \Theta} \tilde{\nabla}_i \dot{Y} + \\ & \overline{\nabla_0 \Theta} (\Delta_h \dot{Y} - \overline{\Gamma}_{0k}^k \dot{\Upsilon} + \frac{1}{6} \dot{X}^\mu \overline{\nabla_\mu R} + \frac{1}{3} \overline{R} \dot{Y}) - \overline{\nabla^k \Theta} (\tilde{\nabla}_k \dot{\Upsilon} - \overline{\Gamma}_{0k}^i \tilde{\nabla}_i \dot{Y}) = 0, \\ (iii) \quad & \overline{A}_{ij} \equiv 2 \nabla_{(i} \dot{X}_{j)} - 2 \dot{Y} \overline{g}_{ij} = 0, \\ & \overline{A}_{0i} \equiv \dot{\Lambda}_i + \nabla_i \dot{X}_0 = 0, \\ & \overline{A}_{00} \equiv 2 \dot{\Lambda}_0 + 2 \dot{Y} = 0, \\ & \overline{\nabla_0 A_{ij}} \equiv 2 \tilde{\nabla}_{(i} \dot{\Lambda}_{j)} - 2 \overline{\Gamma}_{0(i}^k \nabla_{k} \dot{X}_{j)} - 2 \overline{\Gamma}_{ij}^0 \dot{\Lambda}_0 + 2 \overline{R}_{0(ij)}^\mu \dot{X}_\mu - 2 \dot{\Upsilon} \overline{g}_{ij} = 0, \\ (iv) \quad & \overline{B}_{ij} \equiv (\dot{X}^\mu \overline{\nabla_\mu L_{ij}} + 2 \overline{L}_{\mu(i} \nabla_{j)} \dot{X}^\mu + \tilde{\nabla}_i \tilde{\nabla}_j \dot{Y} - \overline{\Gamma}_{ij}^0 \dot{\Upsilon})^\vee = 0, \\ & (\overline{\nabla_0 B_{ij}^{\text{intr}}})^\vee := [\mathcal{L}_{\dot{X}} \overline{\nabla_0 L_{ij}} + 2 \overline{L}_{\mu(i} \partial_{j)} \dot{\Lambda}^\mu + \overline{\Gamma}_{j\alpha}^\mu \dot{\Lambda}^\alpha - \overline{\Gamma}_{0j}^k \nabla_k \dot{X}^\mu) + 2 \overline{L}_{k(i} \overline{R}_{j)\mu 0}^k \dot{X}^\mu + \\ & \tilde{\nabla}_i \tilde{\nabla}_j \dot{\Upsilon} - \overline{\Gamma}_{ij}^0 (\Delta_h \dot{Y} + \frac{1}{6} \dot{X}^\mu \overline{\nabla_\mu R} + \frac{1}{3} \overline{R} \dot{Y}) - 2 \overline{\Gamma}_{0(i}^k \tilde{\nabla}_{j)} \tilde{\nabla}_k \dot{Y} + (\overline{R}_{0ij}^0 + \overline{\Gamma}_{0i}^k \overline{\Gamma}_{jk}^0 + \\ & \overline{\Gamma}_{ij}^0 \overline{\Gamma}_{0k}^k) \dot{\Upsilon} + (\overline{R}_{0ij}^k - \tilde{\nabla}_i \overline{\Gamma}_{0j}^k) \tilde{\nabla}_k \dot{Y}]^\vee = 0. \end{aligned}$$

PROOF: Assume that there exist fields \dot{X} , $\dot{\Lambda}$, \dot{Y} and $\dot{\Upsilon}$ which satisfy (i)-(iv). These fields provide the initial data for the wave equations (3.2) and (3.3) for X and Y . A solution exists due to standard results. Once (3.2) and (3.3) are satisfied the considerations above reveal that (i)-(iv) are equivalent to (a)-(c) of Theorem 3.2, i.e. all the hypotheses of Theorem 3.2 hold and we are done. From the derivation of (i)-(iv) it follows that these conditions are necessary, as well. \square

REMARK 3.4 We call the equations in (i)-(iv) the *(proper) KID equations on \mathcal{H}* .

4 KID equations on a spacelike \mathcal{S}^-

4.1 Derivation of the (reduced) KID equations

Let us restrict now attention to space-times which contain a spacelike \mathcal{S}^- , which we take henceforth as initial surface (recall that this requires a positive

cosmological constant λ). We impose the $(R = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu})$ -wave-map gauge condition introduced in Section 2.3. Recall that the freely prescribable data on \mathcal{I}^- for the Cauchy problem are the conformal class of a Riemannian metric h_{ij} and a symmetric, trace- and divergence-free tensor D_{ij} . The MCFE then imply the constraint equations (2.48)-(2.54) on \mathcal{I}^- . In Appendix A it is shown that a solution to the MCFE further satisfies

$$\overline{\nabla_0 \nabla_0 \Theta} = 0, \quad \bar{R}_{0ij}{}^k = 0. \quad (4.1)$$

We are now ready to evaluate the conditions (i)-(iv) of Theorem 3.3.

The condition (i) becomes

$$\mathring{X}^0 = 0, \quad \mathring{\Lambda}^0 = \mathring{Y}. \quad (4.2)$$

Then condition (ii) is satisfied iff (set $\Delta_{\tilde{g}} := \tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j$)

$$\mathring{\Upsilon} = 0, \quad \mathring{X}^i \tilde{\nabla}_i \tilde{R} + 2\tilde{R} \mathring{Y} + 4\Delta_{\tilde{g}} \mathring{Y} = 0. \quad (4.3)$$

The condition $\bar{A}_{\mu\nu} = 0$ requires

$$\mathring{\Lambda}^i = 0, \quad (4.4)$$

$$\mathring{Y} = \frac{1}{3} \tilde{\nabla}_i \mathring{X}^i, \quad (4.5)$$

$$(\tilde{\nabla}_{(i} \mathring{X}_{j)})^\vee = 0. \quad (4.6)$$

The condition $\overline{\nabla_0 A_{ij}} = 0$ is then automatically fulfilled.

We reconsider the second condition in (4.3). Observe that (4.5), (4.6) and the second Bianchi identity imply the relation

$$\begin{aligned} 0 = \tilde{\nabla}^i \tilde{\nabla}^j \bar{A}_{ij} &= \tilde{\nabla}_i \Delta_{\tilde{g}} \mathring{X}^i + \Delta_{\tilde{g}} \mathring{Y} + \frac{1}{2} \mathring{X}^i \tilde{\nabla}_i \tilde{R} + \underbrace{\tilde{R}_{jk} \tilde{\nabla}^j \mathring{X}^k}_{=\tilde{R} \mathring{Y}} \\ &= 4\Delta_{\tilde{g}} \mathring{Y} + \mathring{X}^i \tilde{\nabla}_i \tilde{R} + 2\tilde{R} \mathring{Y}, \end{aligned}$$

i.e. (4.3) follows from (4.5) and (4.6).

We have

$$\begin{aligned} \check{\bar{B}}_{ij} &= (\mathring{X}^k \tilde{\nabla}_k \tilde{L}_{ij} + 2\tilde{L}_{k(i} \tilde{\nabla}_{j)} \mathring{X}^k + \tilde{\nabla}_i \tilde{\nabla}_j \mathring{Y})^\vee \\ &= \mathcal{L}_{\mathring{X}^k \partial_k} \check{\bar{L}}_{ij} + (\tilde{\nabla}_i \tilde{\nabla}_j \mathring{Y})^\vee, \end{aligned}$$

and

$$\begin{aligned} (\overline{\nabla_0 B_{ij}^{\text{intr}}})^\vee &= -\sqrt{\frac{\lambda}{3}} (D_{ij} \mathring{Y} + \mathring{X}^k \tilde{\nabla}_k D_{ij} + 2D_{k(i} \tilde{\nabla}_{j)} \mathring{X}^k) \\ &= -\sqrt{\frac{\lambda}{3}} (\mathcal{L}_{\mathring{X}^k \partial_k} D_{ij} + D_{ij} \mathring{Y}). \end{aligned}$$

We observe that due to the second Bianchi identity and (4.5)

$$\begin{aligned} \tilde{\nabla}_i \tilde{\nabla}^k \bar{A}_{jk} &= \mathcal{L}_{\mathring{X}^k \partial_k} \tilde{R}_{\mu\nu} + \tilde{\nabla}_i \tilde{\nabla}_j \mathring{Y} + 2\mathring{X}^k \tilde{\nabla}_{[i} \tilde{R}_{j]k} + \Delta_{\tilde{g}} \tilde{\nabla}_i \mathring{X}_j \\ &\quad + 2\tilde{R}_i{}^k{}_j{}^l \tilde{\nabla}_k \mathring{X}_l - 2\tilde{R}_{ij} \mathring{Y} - \tilde{R}_i{}^k \bar{A}_{jk}. \end{aligned}$$

Symmetrizing this expression, taking its traceless part and taking $\bar{A}_{ij} = 0$ into account we end up with

$$\mathcal{L}_{\dot{X}^k \partial_k} \tilde{\tilde{L}}_{\mu\nu} + (\tilde{\nabla}_i \tilde{\nabla}_j \dot{Y})^\vee = 0 ,$$

i.e. $\tilde{\tilde{B}}_{ij}$ holds automatically, as well.

THEOREM 4.1 *Assume we have been given a 3 + 1-dimensional “unphysical” space-time (\mathcal{M}, g, Θ) , with $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ a smooth solution of the MCFE with $\lambda > 0$ in the $(R = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \bar{g}_{\mu\nu})$ -wave-map gauge. Then there exists a smooth vector field X satisfying the unphysical Killing equations (3.1) on $D^+(\mathcal{I}^-)$ (and thus corresponding to a Killing vector field of the physical space-time) if and only if there exists a conformal Killing vector field \dot{X} on $(\mathcal{I}^-, \tilde{g} = h_{ij} dx^i dx^j)$ such that the reduced KID equations*

$$\mathcal{L}_{\dot{X}} D_{ij} + \frac{1}{3} D_{ij} \tilde{\nabla}_k \dot{X}^k = 0 \quad (4.7)$$

hold (recall that the symmetric, trace- and divergence-free tensor field $D_{ij} = \bar{d}_{0i0j}$ belongs to the freely prescribable initial data). In that case X satisfies

$$\bar{X}^0 = 0 , \quad \bar{X}^i = \dot{X}^i , \quad \bar{\nabla}_0 \bar{X}^0 = \frac{1}{3} \tilde{\nabla}_i \dot{X}^i , \quad \bar{\nabla}_0 \bar{X}^i = 0 . \quad (4.8)$$

REMARK 4.2 Note that, in contrast to the $\lambda = 0$ -case treated in [10], the candidate fields, i.e. the conformal Killing fields on \mathcal{I}^- , do depend here on the initial data $h = h_{ij} dx^i dx^j$.

REMARK 4.3 For initial data with $D_{ij} = 0$ the reduced Killing equations (4.7) are always satisfied, and each candidate field, i.e. each conformal Killing field on the initial manifold, extends to a Killing field of the physical space-time.

In terms of an initial value problem Theorem 2.1 and 4.1 state that given a Riemannian manifold (\mathcal{H}, h) and a symmetric, trace- and divergence-free tensor field D_{ij} there exists an (up to isometries) unique evolution into a space-time manifold (\mathcal{M}, g, Θ) with $\mathcal{H} = \mathcal{I}^-$, $\bar{g}_{ij} = h_{ij}$ and $\bar{d}_{0i0j} = D_{ij}$ which fulfills the MCFE and contains a vector field satisfying the unphysical Killing equations (3.1) if and only if there exists a conformal Killing vector field \dot{X} on (\mathcal{H}, h) such that the reduced KID equations (4.7) hold.

4.2 Properties of the reduced KID equations

We compute how the reduced KID equations (4.7) behave under conformal transformations. For this consider the conformally rescaled metric $\tilde{\tilde{g}} := \Omega^2 \tilde{g}$ with Ω some positive function. Expressed in terms of $\tilde{\tilde{g}}$ (4.7) becomes

$$\mathcal{L}_{\dot{X}} (\Omega^{-1} D_{ij}) + \frac{1}{3} (\Omega^{-1} D_{ij}) \tilde{\tilde{\nabla}}_k \dot{X}^k = 0 , \quad (4.9)$$

i.e. they are conformally covariant in the following sense:

LEMMA 4.4 *The pair (\tilde{g}_{ij}, D_{ij}) is a solution of the reduced KID equations (4.7) if and only if the conformally rescaled pair $(\Omega^2 \tilde{g}_{ij}, \Omega^{-1} D_{ij})$, with Ω some positive function, is a solution of these equations.*

This is consistent with the observation that conformal rescalings of the initial data do not change the isometry class of the emerging space-time.

4.3 Some special cases

Let us finish by taking a look at some special cases:

4.3.1 Compact initial manifolds

We consider a *compact* initial manifold $(\mathcal{I}^-, \tilde{g})$ and assume that it admits a conformal Killing field \dot{X} . Then there exists (cf. e.g. [7]) a positive function Ω such that the conformally rescaled metric $\tilde{\tilde{g}} = \Omega^2 \tilde{g}$ has one of the following properties:

- Either $(\mathcal{I}^-, \tilde{\tilde{g}}) = (S^3, s_{ij} dx^i dx^j)$ is the standard 3-sphere,
- or \dot{X} is a Killing vector field w.r.t. $\tilde{\tilde{g}}$.

If $(\mathcal{I}^-, \tilde{\tilde{g}})$ is the round 3-sphere all the conformal Killing fields are explicitly known. In the second case where \dot{X} is a Killing vector field w.r.t. $\tilde{\tilde{g}}$ the equation (4.9) simplifies to

$$\mathcal{L}_{\dot{X}}(\Omega^{-1} D_{ij}) = 0. \quad (4.10)$$

That implies:

LEMMA 4.5 *Consider a solution of the vacuum Einstein equations which admits a compact spacelike \mathcal{I}^- and has a non-trivial Killing field. If $(\mathcal{I}^-, \tilde{g})$ is not conformal to a standard 3-sphere, then there exists a choice of conformal factor so that space-time Killing vector corresponds to a Killing field (rather than a conformal Killing field) of $(\mathcal{I}^-, \tilde{g})$.*

4.3.2 Maximally symmetric space-times

Let us consider the case where the initial manifold admits the maximal number of conformal Killing vector fields. Clearly this is a prerequisite to obtain a maximally symmetric space-time once the evolution problem has been solved. A connected 3-dimensional Riemannian manifold (\mathcal{H}, h) admits at most 10 linearly independent conformal Killing vector fields. If equality is attained, (\mathcal{H}, h) is known to be locally conformally flat [12].

Let us first consider the compact case. We use a classical result due to Kuiper (cf. [7]):

THEOREM 4.6 *For any n -dimensional, simply connected, conformally flat Riemann manifold (\mathcal{H}, h) , there exists a conformal immersion $(\mathcal{H}, h) \hookrightarrow (S^n, s = s_{ij} dx^i dx^j)$, the so-called developing map, which is unique up to composition with Möbius transformations. If \mathcal{H} is compact this map defines a conformal diffeomorphism from (\mathcal{H}, h) onto (S^n, s) .*

Since only the conformal class of the initial manifold matters we thus may assume (\mathcal{H}, h) for compact \mathcal{H} to be the standard 3-sphere from the outset. To end up with a maximally symmetric physical space-time containing 10 independent Killing fields one needs to make sure that each of the conformal Killing fields extends to a space-time vector field satisfying the unphysical Killing equations (3.1). In other words one needs to choose D_{ij} such that the reduced KID equations (4.7) hold for each and every conformal Killing field on (S^3, s) . Via a

stereographic projection onto Euclidean space one shows that this is only possible when D_{ij} is proportional to the round sphere metric. But D_{ij} is traceless, and thus needs to vanish. For data $(\mathcal{H}, h) = (S^3, s)$ and $D_{ij} = 0$ one ends up with de Sitter space-time. This is in accordance with the fact that de Sitter space-time is (up to isometries) the unique maximally symmetric, complete space-time with positive scalar curvature.

The non-compact case is somewhat more involved since the developing map does in general not define a global conformal diffeomorphism into (S^n, s) . For convenience let us therefore make some simplifying assumptions on (\mathcal{H}, h) which allow us to apply a result by Schoen & Yau [11] (we restrict attention to 3 dimensions when stating it):

THEOREM 4.7 *Let (\mathcal{H}, h) be a complete, simply connected, conformally flat 3-dimensional Riemannian manifold and $\Phi : \mathcal{H} \hookrightarrow S^3$ its developing map. Assume that $|R(h)|$ is bounded on \mathcal{H} and that $d(\mathcal{H}) < \frac{1}{3}$.⁴ Then Φ is one-to-one and gives a conformal diffeomorphism from \mathcal{H} onto a simply connected domain of S^3 .*

We conclude, again, that the emerging space-time will be maximally symmetric iff $D_{ij} = 0$, and will be (isometric to) a part of de Sitter space-time.

4.3.3 Non-existence of stationary space-times

For (\mathcal{M}, g, Θ) to contain a timelike isometry there must exist a vector field X satisfying the unphysical Killing equations (3.1) which is null on \mathcal{I}^- (it cannot be timelike since $\overline{X}^0 = 0$),

$$0 = \overline{g}_{\mu\nu} \overline{X}^\mu \overline{X}^\nu = h_{ij} \overline{X}^i \overline{X}^j \implies \overline{X}^i = 0.$$

But then the preceding considerations show that $\overline{X}^\mu = \overline{\nabla_0 X}^\mu = \overline{Y} = \overline{\nabla_0 Y} = 0$, and solving the wave equations for X and Y , (3.2) and (3.3), yields that X vanishes identically. It follows that there is no vacuum space-time with $\lambda > 0$ which is stationary near \mathcal{I}^- . (Compare [4, Section 4].)

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⁴For the definition of the invariant $d(\mathcal{H})$ in terms of the minimal Green's function for the conformal Laplacian we refer the reader to [11].

A Equivalence between the CWE and the MCFE

A.1 Conformal wave equations (CWE)

In [9] the MCFE (2.1)-(2.6) have been rewritten as a system of conformal wave equations (CWE),

$$\square_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa}L_\nu{}^\kappa - g_{\mu\nu}|L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma + \frac{1}{6}\nabla_\mu\nabla_\nu R, \quad (\text{A.1})$$

$$\square_g s = \Theta|L|^2 - \frac{1}{6}\nabla_\kappa R \nabla^\kappa \Theta - \frac{1}{6}sR, \quad (\text{A.2})$$

$$\square_g \Theta = 4s - \frac{1}{6}\Theta R, \quad (\text{A.3})$$

$$\square_g^{(H)} d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa + \frac{1}{2}R d_{\mu\nu\sigma\rho}, \quad (\text{A.4})$$

$$R_{\mu\nu}^{(H)}[g] = 2L_{\mu\nu} + \frac{1}{6}Rg_{\mu\nu}. \quad (\text{A.5})$$

Here

$$R_{\mu\nu}^{(H)} := R_{\mu\nu} - g_{\sigma(\mu}\hat{\nabla}_{\nu)}H^\sigma, \quad (\text{A.6})$$

denotes the reduced Ricci tensor. The reduced wave-operator $\square_g^{(H)}$ (which is needed to obtain a PDE-system with a diagonal principal part) is defined via its action on covector fields v_λ ,

$$\square_g^{(H)} v_\lambda := \square_g v_\lambda - g_{\sigma[\lambda}(\hat{\nabla}_{\mu]}H^\sigma)v^\mu + (2L_{\mu\lambda} - R_{\mu\lambda}^{(H)} + \frac{1}{6}Rg_{\mu\lambda})v^\mu, \quad (\text{A.7})$$

and similar formulae hold for higher-valence covariant tensor fields.

In the following we want to show that a solution of the CWE in the gauge (2.25) is a solution of the MCFE if and only if the constraint equations (2.47)-(2.54) hold on \mathcal{S}^- ,

$$h^{ij}D_{ij} = 0, \quad \tilde{\nabla}^j D_{ij} = 0, \quad (\text{A.8})$$

$$\bar{g}_{00} = -1, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = h_{ij}, \quad \overline{\partial_0 g_{\mu\nu}} = 0, \quad (\text{A.9})$$

$$\bar{\Theta} = 0, \quad \overline{\partial_0 \Theta} = \sqrt{\frac{\lambda}{3}}, \quad (\text{A.10})$$

$$\bar{s} = 0, \quad \overline{\partial_0 s} = \sqrt{\frac{\lambda}{48}}\tilde{R}, \quad (\text{A.11})$$

$$\bar{L}_{ij} = \tilde{L}_{ij}, \quad \bar{L}_{0i} = 0, \quad \bar{L}_{00} = \frac{1}{4}\tilde{R}, \quad (\text{A.12})$$

$$\overline{\partial_0 L_{ij}} = -\sqrt{\frac{\lambda}{3}}D_{ij}, \quad \overline{\partial_0 L_{0i}} = \frac{1}{4}\tilde{\nabla}_i \tilde{R}, \quad \overline{\partial_0 L_{00}} = 0, \quad (\text{A.13})$$

$$\bar{d}_{0i0j} = D_{ij}, \quad \bar{d}_{0ijk} = \sqrt{\frac{3}{\lambda}}\tilde{C}_{ijk}, \quad (\text{A.14})$$

$$\overline{\partial_0 d_{0i0j}} = \sqrt{\frac{3}{\lambda}}\tilde{B}_{ij}, \quad \overline{\partial_0 d_{0ijk}} = 2\tilde{\nabla}_{[j}D_{k]i}. \quad (\text{A.15})$$

A.2 An intermediate result

In close analogy to [9, Theorem 3.7] one establishes the following result:

THEOREM A.1 *Assume we have been given data $(\hat{g}_{\mu\nu}, \hat{K}_{\mu\nu}, \hat{s}, \hat{S}, \hat{\Theta}, \hat{\Omega}, \hat{L}_{\mu\nu}, \hat{M}_{\mu\nu}, \hat{d}_{\mu\nu\sigma}{}^\rho, \hat{D}_{\mu\nu\sigma}{}^\rho)$ on a spacelike hypersurface \mathcal{H} and a gauge source function*

R , such that $\dot{g}_{\mu\nu}$ is the restriction to \mathcal{H} of a Lorentzian metric, $\dot{K}_{\mu\nu}$, $\dot{L}_{\mu\nu}$ and $\dot{M}_{\mu\nu}$ are symmetric, $\dot{L} \equiv \dot{L}_\mu{}^\mu = \overline{R}/6$, $\dot{M}_\mu{}^\mu = \overline{\partial_0 R}/6$, and such that $\dot{d}_{\mu\nu\sigma}{}^\rho$ and $\dot{D}_{\mu\nu\sigma}{}^\rho$ satisfy all the algebraic properties of the Weyl tensor. Suppose further that there exists a solution $(g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ of the CWE (A.1)-(A.5) with gauge source function R which induces the above data on \mathcal{H} ,

$$\begin{aligned} \overline{g}_{\mu\nu} &= \dot{g}_{\mu\nu}, \quad \overline{s} = \dot{s}, \quad \overline{\Theta} = \dot{\Theta}, \quad \overline{L}_{\mu\nu} = \dot{L}_{\mu\nu}, \quad \overline{d}_{\mu\nu\sigma}{}^\rho = \dot{d}_{\mu\nu\sigma}{}^\rho, \\ \overline{\partial_0 g}_{\mu\nu} &= \dot{K}_{\mu\nu}, \quad \overline{\partial_0 s} = \dot{S}, \quad \overline{\partial_0 \Theta} = \dot{\Omega}, \quad \overline{\partial_0 L}_{\mu\nu} = \dot{M}_{\mu\nu}, \quad \overline{\partial_0 d}_{\mu\nu\sigma}{}^\rho = \dot{D}_{\mu\nu\sigma}{}^\rho, \end{aligned}$$

and fulfills the following conditions:

1. The MCFE (2.1)-(2.4) and their covariant derivatives are fulfilled on \mathcal{H} ;
2. equation (2.5) holds at one point on \mathcal{H} ;
3. $\overline{W}_{\mu\nu\sigma}{}^\rho[g] = \overline{\Theta} \overline{d}_{\mu\nu\sigma}{}^\rho$ and $\overline{\nabla_0 W}_{\mu\nu\sigma}{}^\rho[g] = \overline{\nabla_0(\Theta d_{\mu\nu\sigma}{}^\rho)}$;
4. the wave-gauge vector H^σ and its first- and second-order covariant derivatives $\nabla_\mu H^\sigma$ and $\nabla_\mu \nabla_\nu H^\sigma$ vanish on \mathcal{H} ;
5. the covector field $\zeta_\mu \equiv -4(\nabla_\nu L_\mu{}^\nu - \frac{1}{6}\nabla_\mu R)$ and its covariant derivative $\nabla_\nu \zeta_\mu$ vanish on \mathcal{H} .

Then

- a) $H^\sigma = 0$ and $R_g = R$ (where R_g denotes the Ricci scalar of $g_{\mu\nu}$);
- b) $L_{\mu\nu}$ is the Schouten tensor of $g_{\mu\nu}$;
- c) $\Theta d_{\mu\nu\sigma}{}^\rho$ is the Weyl tensor of $g_{\mu\nu}$;
- d) $(g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ solves the MCFE (2.1)-(2.6) in the $(H^\sigma = 0, R_g = R)$ -gauge.

The conditions 1.-5. are also necessary for d) to be true.

A.3 Applicability of Theorem A.1 on \mathcal{J}^-

We now consider the case where $\mathcal{H} = \mathcal{J}^-$. Using the gauge (2.25) we want to show that the hypotheses of Theorem A.1 are fulfilled by any tuple $(g_{\mu\nu}, s, \Theta, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ which satisfies the constraint equations (A.8)-(A.15) and the CWE.

For $R = 0$ the CWE reduce to

$$\square_g^{(H)} L_{\mu\nu} = 4L_{\mu\kappa} L_\nu{}^\kappa - g_{\mu\nu} |L|^2 - 2\Theta d_{\mu\sigma\nu}{}^\rho L_\rho{}^\sigma, \quad (\text{A.16})$$

$$\square_g s = \Theta |L|^2, \quad (\text{A.17})$$

$$\square_g \Theta = 4s, \quad (\text{A.18})$$

$$\square_g^{(H)} d_{\mu\nu\sigma\rho} = \Theta d_{\mu\nu\kappa}{}^\alpha d_{\sigma\rho\alpha}{}^\kappa - 4\Theta d_{\sigma\kappa[\mu}{}^\alpha d_{\nu]\alpha\rho}{}^\kappa, \quad (\text{A.19})$$

$$R_{\mu\nu}^{(H)}[g] = 2L_{\mu\nu}. \quad (\text{A.20})$$

First of all note that $\overline{L} = 0 = \overline{R}/6$ and $\overline{\partial_0 L} = 0 = \overline{\partial_0 R}/6$, as required. Moreover (A.10) implies that (2.5) is satisfied on \mathcal{J}^- , i.e. it remains to verify that the hypotheses 1. and 3.-5. in Theorem A.1 are fulfilled.

Recall that in our gauge the only non-vanishing Christoffel symbols on \mathcal{J}^- are $\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ij}^k$, and that this implies that the action of ∇_0 and ∂_0 as well as the action of ∇_i and $\tilde{\nabla}_i$ coincides on \mathcal{J}^- .

A.3.1 Vanishing of \overline{H} , $\overline{\nabla H}$ and $\overline{\nabla\nabla H}$

We have

$$\overline{H}^0 \equiv \overline{g}^{\mu\nu}(\overline{\Gamma}_{\mu\nu}^0 - \hat{\Gamma}_{\mu\nu}^0) = 0, \quad (\text{A.21})$$

$$\overline{H}^i \equiv \overline{g}^{\mu\nu}(\overline{\Gamma}_{\mu\nu}^i - \hat{\Gamma}_{\mu\nu}^i) = 0. \quad (\text{A.22})$$

Equation (A.20) can be written as

$$R_{\mu\nu} - g_{\sigma(\mu} \hat{\nabla}_{\nu)} H^\sigma = 2L_{\mu\nu}. \quad (\text{A.23})$$

Invoking $\overline{H}^\sigma = 0$ that gives

$$\begin{aligned} \overline{R}_{00} + \overline{\partial_0 H^0} &= 2\overline{L}_{00}, \\ \overline{R}_{0i} - \frac{1}{2}\overline{g_{ij}\partial_0 H^j} &= 2\overline{L}_{0i}, \\ \overline{R}_{ij} &= 2\overline{L}_{ij}. \end{aligned}$$

On the other hand, with (A.9) we find

$$\begin{aligned} \overline{R}_{00} &= -\overline{\partial_0 \Gamma_{0k}^k} = -\frac{1}{2}\overline{g^{kl}\partial_0 \partial_0 g_{kl}}, \\ \overline{R}_{0i} &= -\overline{\partial_0 \Gamma_{ik}^k} = 0, \\ \overline{R}_{ij} &= \overline{\partial_0 \Gamma_{ij}^0} + \tilde{R}_{ij} = \frac{1}{2}\overline{\partial_0 \partial_0 g_{ij}} + \tilde{R}_{ij}. \end{aligned}$$

Taking (A.12) into account, we conclude that

$$\overline{\partial_0 \partial_0 g_{ij}} = 2\tilde{R}_{ij} - \overline{g_{ij}\tilde{R}}, \quad (\text{A.24})$$

as well as $\overline{\partial_0 H^\sigma} = 0$, and we end up with

$$\overline{\nabla_\mu H^\sigma} = 0. \quad (\text{A.25})$$

Note that this implies

$$0 = \overline{\partial_0 H^0} = \overline{g^{\mu\nu}\partial_0 \Gamma_{\mu\nu}^0} = \frac{1}{2}\overline{\partial_0 \partial_0 g_{00}} - \frac{1}{2}\tilde{R}, \quad (\text{A.26})$$

$$0 = \overline{\partial_0 H^k} = \overline{g^{\mu\nu}\partial_0 \Gamma_{\mu\nu}^k} = -\overline{g^{kl}\partial_0 \partial_0 g_{0l}}, \quad (\text{A.27})$$

i.e.

$$\overline{\partial_0 \partial_0 g_{00}} = \tilde{R}, \quad \overline{\partial_0 \partial_0 g_{0i}} = 0. \quad (\text{A.28})$$

We give a list of the transverse derivatives of the Christoffel symbols on \mathcal{S}^- ,

$$\overline{\partial_0 \Gamma_{00}^0} = -\frac{1}{2}\tilde{R}, \quad \overline{\partial_0 \Gamma_{ij}^0} = \overline{g_{jk}\partial_0 \Gamma_{0i}^k} = \tilde{R}_{ij} - \frac{1}{2}\overline{g_{ij}\tilde{R}}, \quad (\text{A.29})$$

$$\overline{\partial_0 \Gamma_{0i}^0} = \overline{\partial_0 \Gamma_{00}^k} = \overline{\partial_0 \Gamma_{ij}^k} = 0. \quad (\text{A.30})$$

Using (A.23) that yields with $\overline{H}^\sigma = 0 = \overline{\nabla_\mu H^\sigma}$ the relation

$$\overline{\partial_0 R_{\mu\nu}} - \overline{g_{\sigma(\mu}\partial_{\nu)}\partial_0 H^\sigma} = 2\overline{\partial_0 L_{\mu\nu}}, \quad (\text{A.31})$$

and thus

$$\begin{aligned}\overline{\partial_0 R_{00}} + \overline{\partial_0 \partial_0 H^0} &= 2\overline{\partial_0 L_{00}} , \\ \overline{\partial_0 R_{0i}} - \frac{1}{2}\overline{g_{ij}\partial_0 \partial_0 H^j} &= 2\overline{\partial_0 L_{0i}} , \\ \overline{\partial_0 R_{ij}} &= 2\overline{\partial_0 L_{ij}} .\end{aligned}$$

We compute

$$\begin{aligned}\overline{\partial_0 R_{00}} &= -\overline{\partial_0 \partial_0 \Gamma_{0k}^k} = -\frac{1}{2}\overline{g^{kl}\partial_0 \partial_0 \partial_0 g_{kl}} , \\ \overline{\partial_0 R_{0i}} &= \underbrace{\tilde{\nabla}_k \overline{\partial_0 \Gamma_{0i}^k}}_{=0} - \overline{\partial_0 \partial_0 \Gamma_{ik}^k} = -\frac{1}{2}\tilde{\nabla}_i (\overline{g^{kl}\partial_0 \partial_0 g_{kl}}) = \frac{1}{2}\tilde{\nabla}_i \tilde{R} , \\ \overline{\partial_0 R_{ij}} &= \overline{\partial_0 \partial_0 \Gamma_{ij}^0} = \frac{1}{2}\overline{\partial_0 \partial_0 \partial_0 g_{ij}} .\end{aligned}$$

From (A.13) we deduce that

$$\overline{\partial_0 \partial_0 \partial_0 g_{ij}} = -4\sqrt{\frac{\lambda}{3}}D_{ij} , \quad (\text{A.32})$$

from which we obtain $\overline{\partial_0 \partial_0 H^\sigma} = 0$, and thus

$$\overline{\nabla_\mu \nabla_\nu H^\sigma} = 0 . \quad (\text{A.33})$$

A.3.2 Vanishing of $\bar{\zeta}$ and $\overline{\nabla \zeta}$

In our gauge we have

$$\zeta_\mu = -4\nabla_\alpha L_\mu{}^\alpha . \quad (\text{A.34})$$

We invoke (A.12) and (A.13) to obtain

$$\bar{\zeta}_0 = 4\overline{\partial_0 L_{00}} - 4\overline{g^{kl}\nabla_k \bar{L}_{0l}} = 0 , \quad (\text{A.35})$$

$$\bar{\zeta}_i = 4\overline{\partial_0 L_{0i}} - 4\overline{g^{kl}\tilde{\nabla}_k \bar{L}_{il}} = 0 . \quad (\text{A.36})$$

The computation of $\overline{\nabla_0 \zeta_\mu}$ requires the knowledge of certain second-order transverse derivatives of $\bar{L}_{\mu\nu}$ which we compute from the CWE (A.16). Since $\bar{H}^\sigma = 0 = \overline{\nabla_\mu H^\sigma}$ we have

$$\begin{aligned}\overline{\square_g L_{\mu\nu}} &= \overline{\square_g^{(H)} L_{\mu\nu}} = 4\overline{L_{\mu\kappa} \bar{L}_\nu{}^\kappa} - \overline{g_{\mu\nu} |\bar{L}|^2} \iff \\ \overline{\nabla_0 \nabla_0 L_{\mu\nu}} &= \Delta_{\bar{g}} \overline{L_{\mu\nu}} - 4\overline{L_{\mu\kappa} \bar{L}_\nu{}^\kappa} + \overline{g_{\mu\nu} [\bar{L}_k{}^l \bar{L}_l{}^k + (\bar{L}_{00})^2]} ,\end{aligned}$$

whence

$$\overline{\nabla_0 \nabla_0 L_{00}} = \frac{1}{4}\Delta_{\bar{g}} \tilde{R} - |\tilde{R}|^2 + \frac{1}{2}\tilde{R}^2 , \quad (\text{A.37})$$

$$\overline{\nabla_0 \nabla_0 L_{0i}} = 0 . \quad (\text{A.38})$$

From

$$\overline{\nabla_0 \zeta_\mu} = 4\overline{\nabla_0 \nabla_0 L_{0\mu}} - 4\overline{g^{kl}\tilde{\nabla}_k \nabla_0 \bar{L}_{l\mu}} - 4\overline{R_{0k\mu}{}^l \bar{L}_l{}^k} - 4\overline{R_{00} \bar{L}_{0\mu}}$$

and

$$\overline{R}_{0k0}{}^l = -\tilde{R}_k{}^l + \frac{1}{2}\delta_k{}^l \tilde{R} , \quad (\text{A.39})$$

$$\overline{R}_{0ki}{}^l = 0 , \quad (\text{A.40})$$

we conclude that

$$\overline{\nabla_0 \zeta_0} = \frac{3}{2}\tilde{R}^2 - 4|\tilde{R}|^2 - 4\overline{R}_{0k0}{}^l \tilde{L}_l{}^k = 0 , \quad (\text{A.41})$$

$$\overline{\nabla_0 \zeta_i} = 4\sqrt{\frac{\lambda}{3}}\tilde{\nabla}^j D_{ij} - 4\overline{R}_{0ki}{}^l \tilde{L}_l{}^k = 0 . \quad (\text{A.42})$$

A.3.3 Validity of the MCFE (2.1)-(2.4) and their transverse derivatives on \mathcal{J}^-

The independent components of $\overline{\nabla_\rho d_{\mu\nu\sigma}{}^\rho}$, which is antisymmetric in its first two indices, trace-free and satisfies the first Bianchi identity, are

$$\overline{\nabla_\rho d_{ijk}{}^\rho} \quad \text{and} \quad \overline{\nabla_\rho d_{0ij}{}^\rho}$$

(similarly for its transverse derivatives).

It follows from (2.44), (A.14), (A.15) and (A.8) that

$$\overline{\nabla_\rho d_{ijk}{}^\rho} = \tilde{\nabla}^l \overline{d_{ijkl}} - \overline{\nabla_0 d_{0kji}} = 2\tilde{g}_{k[i} \tilde{\nabla}^l D_{j]l} = 0 , \quad (\text{A.43})$$

$$\overline{\nabla_\rho d_{0ij}{}^\rho} = \nabla^k \overline{d_{0ijk}} + \overline{\nabla_0 d_{0i0j}} = 0 . \quad (\text{A.44})$$

We consider the corresponding transverse derivatives. With (A.39) and (A.40) we find

$$\begin{aligned} \overline{\nabla_0 \nabla_\rho d_{ijk}{}^\rho} &= \overline{\nabla_0 \nabla_0 d_{0kij}} + \tilde{\nabla}^l \overline{\nabla_0 d_{ijkl}} - 2\overline{R}_{0[j0}{}^l \overline{d_{0]i}{}_{kl}} + \overline{R}_{0k0}{}^l \overline{d_{0lij}} + \overline{R}_{00} \overline{d_{0kji}} , \\ \overline{\nabla_0 \nabla_\rho d_{0ij}{}^\rho} &= \overline{\nabla_0 \nabla_0 d_{0i0j}} + \tilde{\nabla}^k \overline{\nabla_0 d_{0ijk}} - \overline{R}_{00}{}^k{}^l \overline{d_{ikjl}} + \overline{R}_{0j0}{}^k D_{ik} - \overline{R}_{00} D_{ij} . \end{aligned}$$

The second-order transverse derivatives of the rescaled Weyl tensor follow from the CWE (A.19),

$$\overline{\square_g d_{\mu\nu\sigma\rho}} = \overline{\square_g^{(H)} d_{\mu\nu\sigma\rho}} = 0 \iff \overline{\nabla_0 \nabla_0 d_{\mu\nu\sigma\rho}} = \Delta_{\tilde{g}} \overline{d_{\mu\nu\sigma\rho}} ,$$

hence

$$\overline{\nabla_0 \nabla_0 d_{0ijk}} = \Delta_{\tilde{g}} \overline{d_{0ijk}} = \sqrt{\frac{3}{\lambda}} \Delta_{\tilde{g}} \tilde{C}_{ijk} , \quad (\text{A.45})$$

$$\overline{\nabla_0 \nabla_0 d_{0i0j}} = \Delta_{\tilde{g}} \overline{d_{0i0j}} = \Delta_{\tilde{g}} \overline{D_{ij}} . \quad (\text{A.46})$$

The Bianchi identities together with the identity

$$\tilde{R}_{ijkl} \equiv 2\tilde{g}_{i[k} \tilde{R}_{l]j} - 2\tilde{g}_{j[k} \tilde{R}_{l]i} - \tilde{R} \tilde{g}_{i[k} \tilde{g}_{l]j} , \quad (\text{A.47})$$

which holds in 3 dimensions, imply the following relations for Cotton and Bach tensor,

$$\begin{aligned} \tilde{C}_{[ijk]} &= \tilde{C}^j{}_{ij} = \tilde{\nabla}^k \tilde{C}_{kij} = 0 , \\ \tilde{\nabla}_{[i} \tilde{C}_{j]kl} &= \tilde{\nabla}_{[l} \tilde{C}_{k]ji} + \tilde{R}_{ij[l}{}^m \tilde{L}_{k]m} + \tilde{R}_{kl[i}{}^m \tilde{L}_{j]m} , \\ \tilde{\nabla}^j \tilde{B}_{ij} &= \tilde{R}^{kl} \tilde{C}_{kli} , \\ \tilde{\nabla}_{[i} \tilde{B}_{j]k} &= -\frac{1}{2} \Delta_{\tilde{g}} \tilde{C}_{kji} + \tilde{R}_{[j}{}^l \tilde{C}_{i]kl} - \frac{1}{2} \tilde{R}_k{}^l \tilde{C}_{lij} - g_{k[i} \tilde{C}^l{}_{j]}{}^m \tilde{R}_{lm} + \frac{1}{4} \tilde{R} \tilde{C}_{kji} . \end{aligned}$$

With (2.44), (A.8), (A.14) and (A.15) we then obtain

$$\begin{aligned}\overline{\nabla_0 \nabla_\rho d_{ijk}{}^\rho} &= \sqrt{\frac{3}{\lambda}} \left(2\bar{g}_{k[i} \tilde{\nabla}^l \tilde{B}_{j]l} - 2\tilde{\nabla}_{[i} \tilde{B}_{j]k} - (\Delta_{\tilde{g}} - \frac{\tilde{R}}{2}) \tilde{C}_{kji} + 2\tilde{R}_{[j}{}^l \tilde{C}_{i]kl} - \tilde{R}_k{}^l \tilde{C}_{lij} \right) \\ &= 0, \\ \overline{\nabla_0 \nabla_\rho d_{0ij}{}^\rho} &= 0.\end{aligned}$$

Set

$$\Xi_{\mu\nu} := \nabla_\mu \nabla_\nu \Theta + \Theta L_{\mu\nu} - s g_{\mu\nu}. \quad (\text{A.48})$$

To compute Ξ_{00} we need to know the value of $\overline{\nabla_0 \nabla_0 \Theta}$ which can be determined from the CWE (A.18),

$$\overline{\square_g \Theta} = 4\bar{s} \iff \overline{\nabla_0 \nabla_0 \Theta} = 0. \quad (\text{A.49})$$

Invoking (A.10)-(A.11) we then find

$$\Xi_{ij} = 0, \quad (\text{A.50})$$

$$\Xi_{0i} = \overline{\nabla_i \nabla_0 \Theta} = 0, \quad (\text{A.51})$$

$$\Xi_{00} = \overline{\nabla_0 \nabla_0 \Theta} = 0. \quad (\text{A.52})$$

To calculate the transverse derivative of $\Xi_{\mu\nu}$ on \mathcal{I}^- we need to determine the third-order transverse derivative of Θ

$$\overline{\nabla_0 \square_g \Theta} = 4\overline{\nabla_0 s} \iff \overline{\nabla_0 \nabla_0 \nabla_0 \Theta} = -\sqrt{\frac{\lambda}{12}} \tilde{R}. \quad (\text{A.53})$$

One then straightforwardly verifies with (A.39) and the constraint equations

$$\begin{aligned}\overline{\nabla_0 \Xi_{ij}} &= \overline{\nabla_i \nabla_j \nabla_0 \Theta} + \overline{R_{0i0j} \nabla_0 \Theta} + \overline{L_{ij} \nabla_0 \Theta} - \overline{\nabla_0 s} \bar{g}_{ij} = 0, \\ \overline{\nabla_0 \Xi_{0i}} &= \overline{\nabla_i \nabla_0 \nabla_0 \Theta} + \overline{L_{0i} \nabla_0 \Theta} = 0, \\ \overline{\nabla_0 \Xi_{00}} &= \overline{\nabla_0 \nabla_0 \nabla_0 \Theta} + \overline{L_{00} \nabla_0 \Theta} + \overline{\nabla_0 s} = 0.\end{aligned}$$

Set

$$\Upsilon_\mu := \nabla_\mu s + L_{\mu\nu} \nabla^\nu \Theta. \quad (\text{A.54})$$

We observe that by (A.10)-(A.12)

$$\overline{\Upsilon_0} = \overline{\nabla_0 s} - \overline{L_{00} \nabla_0 \Theta} = 0, \quad (\text{A.55})$$

$$\overline{\Upsilon_i} = 0. \quad (\text{A.56})$$

To compute the corresponding transverse derivatives on \mathcal{I}^- we first of all need to calculate $\overline{\nabla_0 \nabla_0 s}$, which follows from (A.17),

$$\overline{\square_g s} = 0 \iff \overline{\nabla_0 \nabla_0 s} = 0; . \quad (\text{A.57})$$

Employing further the constraint equations and (A.49) we then deduce

$$\begin{aligned}\overline{\nabla_0 \Upsilon_0} &= \overline{\nabla_0 \nabla_0 s} - \overline{\nabla_0 L_{00} \nabla_0 \Theta} - \overline{L_{00} \nabla_0 \nabla_0 \Theta} = 0, \\ \overline{\nabla_0 \Upsilon_i} &= \overline{\nabla_i \nabla_0 s} - \overline{\nabla_0 L_{0i} \nabla_0 \Theta} + \overline{L_{ij} \nabla^j \nabla_0 \Theta} = 0.\end{aligned}$$

Set

$$\varkappa_{\mu\nu\sigma} := 2\nabla_{[\sigma} L_{\nu]\mu} - \nabla_\rho \Theta d_{\nu\sigma\mu}{}^\rho. \quad (\text{A.58})$$

Due to the symmetries $\varkappa_{\mu(\nu\sigma)} = 0$, $\varkappa_{[\mu\nu\sigma]} = 0$ and $\varkappa_{\nu\mu}{}^\nu = 0$ (since $\bar{\zeta}_\mu = 0$ and $L = 0$) its independent components on the initial surface are

$$\overline{\varkappa}_{ijk} \quad \text{and} \quad \overline{\varkappa}_{ij0}.$$

Since also $\overline{\nabla_0 \zeta_\mu} = 0$ an analogous statement holds true for $\overline{\nabla_0 \varkappa_{\mu\nu\sigma}}$. We find with (A.10) and (A.12)-(A.14)

$$\overline{\varkappa}_{ijk} = 2\nabla_{[k}\overline{L_{j]}i} - \overline{\nabla_0 \Theta} \overline{d_{0ijk}} = 0, \quad (\text{A.59})$$

$$\overline{\varkappa}_{ij0} = 2\nabla_{[0}\overline{L_{j]}i} + \overline{\nabla_0 \Theta} \overline{d_{0i0j}} = 0. \quad (\text{A.60})$$

Before we proceed let us first determine the second-order transverse derivative of L_{ij} on \mathcal{S}^- . From the CWE (A.16) we obtain

$$\begin{aligned} \overline{\square_g L_{ij}} &= \overline{\square_g^{(H)} L_{ij} 4\tilde{L}_{ik}\tilde{L}_j{}^k - \tilde{g}_{ij}|\tilde{L}|^2 - \tilde{g}_{ij}(\overline{L_{00}})^2} \iff \\ \overline{\nabla_0 \nabla_0 L_{ij}} &= \Delta_{\tilde{g}} \tilde{L}_{ij} - 4\tilde{L}_{ik}\tilde{L}_j{}^k + \tilde{g}_{ij}(|\tilde{L}|^2 + \frac{1}{16}\tilde{R}^2). \end{aligned} \quad (\text{A.61})$$

For the transverse derivatives we then find with (A.39), (A.40) and (A.49) and the constraint equations

$$\begin{aligned} \overline{\nabla_0 \varkappa_{ijk}} &= 2\tilde{\nabla}_{[k}\overline{\nabla_{|0|} L_{j]}i} - \overline{\nabla_0 \Theta} \overline{\nabla_0 d_{0ijk}} = 0, \\ \overline{\nabla_0 \varkappa_{ij0}} &= \overline{\nabla_0 \nabla_0 L_{ij}} - \tilde{\nabla}_j \overline{\nabla_0 L_{0i}} - \overline{R_{0j0}}{}^k \overline{L_{ik}} - \overline{R_{0i0j}} \overline{L_{00}} + \overline{\nabla_0 \Theta} \overline{\nabla_0 d_{0i0j}} \\ &= \overline{\nabla_0 \nabla_0 L_{ij}} - \frac{1}{4}\tilde{\nabla}_i \tilde{\nabla}_j \tilde{R} + \tilde{L}_j{}^k \tilde{L}_{ik} - \frac{1}{16}\tilde{R}^2 \tilde{g}_{ij} + \tilde{B}_{ij} = 0, \end{aligned}$$

where we have used that

$$\tilde{B}_{ij} = -\Delta_{\tilde{g}} \tilde{L}_{ij} + \frac{1}{4}\tilde{\nabla}_i \tilde{\nabla}_j \tilde{R} - \tilde{g}_{ij}|\tilde{L}|^2 + 3\tilde{L}_{ik}\tilde{L}_j{}^k. \quad (\text{A.62})$$

A.3.4 Vanishing of $\overline{W}_{\mu\nu\sigma}{}^\rho - \overline{\Theta} \overline{d_{\mu\nu\sigma}}{}^\rho$ and $\overline{\nabla_0(W_{\mu\nu\sigma}{}^\rho - \Theta d_{\mu\nu\sigma}{}^\rho)}$

The independent components of the conformal Weyl tensor in adapted coordinates are

$$\overline{W}_{0ij}{}^k \quad \text{and} \quad \overline{W}_{0i0}{}^j.$$

Using the definition of the Weyl tensor

$$W_{\mu\nu\sigma}{}^\rho \equiv R_{\mu\nu\sigma}{}^\rho - 2(g_{\sigma[\mu}L_{\nu]}{}^\rho - \delta_{[\mu}{}^\rho L_{\nu]\sigma})$$

we observe that by (A.39), (A.40) and (A.12) we have

$$\begin{aligned} \overline{W}_{0ij}{}^k &= \tilde{g}_{ij}\overline{L_0}{}^k - \delta_i{}^k \overline{L_{0j}} = 0, \\ \overline{W}_{0i0}{}^j &= \overline{R_{0i0}}{}^j + \overline{L_i}{}^j - \delta_i{}^j \overline{L_{00}} = 0. \end{aligned}$$

To derive expressions for the transverse derivatives recall the formulae (2.35), (A.29)-(A.30) for the Christoffel symbols and their transverse derivatives on \mathcal{S}^- . Since, by (A.24), (A.27) and (A.32), we further have

$$\begin{aligned} \overline{\partial_0 \partial_0 \Gamma_{ij}^k} &= \frac{1}{2}\tilde{g}^{kl}(\tilde{\nabla}_i \overline{\partial_0 \partial_0 g_{jl}} + \tilde{\nabla}_j \overline{\partial_0 \partial_0 g_{il}} - \tilde{\nabla}_l \overline{\partial_0 \partial_0 g_{ij}}) \\ &= 2\tilde{\nabla}_{(i}\tilde{R}_{j)}{}^k - \delta_{(i}{}^k \tilde{\nabla}_{j)}\tilde{R} - \tilde{\nabla}^k \tilde{R}_{ij} + \frac{1}{2}\tilde{g}_{ij}\tilde{\nabla}^k \tilde{R}, \\ \overline{\partial_0 \partial_0 \Gamma_{i0}^j} &= \frac{1}{2}\tilde{g}^{jk}(\partial_0 \partial_0 \partial_0 g_{ik} + \tilde{\nabla}_i \overline{\partial_0 \partial_0 g_{0k}} - \tilde{\nabla}_k \overline{\partial_0 \partial_0 g_{0i}}) = -2\sqrt{\frac{\lambda}{3}}D_i{}^j, \end{aligned}$$

we find that

$$\begin{aligned}
\overline{\nabla_0 R_{0ij}{}^k} &= \overline{\partial_0 R_{0ij}{}^k} = \tilde{\nabla}_i \overline{\partial_0 \Gamma_{0j}^k} - \overline{\partial_0 \partial_0 \Gamma_{ij}^k} \\
&= -\tilde{\nabla}_j \tilde{R}_i{}^k + \frac{1}{2} \delta_i{}^k \tilde{\nabla}_j \tilde{R} + \tilde{\nabla}^k \tilde{R}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{\nabla}^k \tilde{R}, \\
\overline{\nabla_0 R_{0i0}{}^j} &= \overline{\partial_0 R_{0i0}{}^j} = -\overline{\partial_0 \partial_0 \Gamma_{i0}^j} = 2\sqrt{\frac{\lambda}{3}} D_i{}^j.
\end{aligned}$$

Hence

$$\begin{aligned}
\overline{\nabla_0 W_{0ij}{}^k} &= \overline{\nabla_0 R_{0ij}{}^k} + \tilde{g}_{ij} \overline{\nabla_0 L_0{}^k} - \delta_i{}^k \overline{\nabla_0 L_{0j}} = \tilde{\nabla}^k \tilde{L}_{ij} - \tilde{\nabla}_j \tilde{L}_i{}^k = \tilde{C}_{ij}{}^k, \\
\overline{\nabla_0 W_{0i0}{}^j} &= \overline{\nabla_0 R_{0i0}{}^j} + \overline{\nabla_0 L_i{}^j} - \delta_i{}^j \overline{\nabla_0 L_{00}} = \sqrt{\frac{\lambda}{3}} D_i{}^j,
\end{aligned}$$

and we end up with

$$\begin{aligned}
\overline{\nabla_0 (W_{0ij}{}^k - \Theta d_{0ij}{}^k)} &= \overline{\nabla_0 W_{0ij}{}^k} - \overline{\nabla_0 \Theta} \overline{d_{0ij}{}^k} = 0, \\
\overline{\nabla_0 (W_{0i0}{}^j - \Theta d_{0i0}{}^j)} &= \overline{\nabla_0 W_{0i0}{}^j} - \overline{\nabla_0 \Theta} \overline{d_{0i0}{}^j} = 0,
\end{aligned}$$

which completes the proof that Theorem A.1 is applicable supposing that the initial data for the CWE satisfy the constraint equations (A.8)-(A.15) on \mathcal{I}^- .

THEOREM A.2 *Let us suppose we have been given a Riemannian metric h_{ij} and a smooth tensor field D_{ij} on \mathcal{I}^- . A smooth solution $(g_{\mu\nu}, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho, \Theta, s)$ of the CWE (A.16)-(A.20) to the future of \mathcal{I}^- with initial data*

$$\begin{aligned}
(\bar{g}_{\mu\nu} = \mathring{g}_{\mu\nu}, \quad \overline{\partial_0 g_{\mu\nu}} = \mathring{K}_{\mu\nu}, \quad \bar{L}_{\mu\nu} = \mathring{L}_{\mu\nu}, \quad \overline{\partial_0 L_{\mu\nu}} = \mathring{M}_{\mu\nu}, \quad \bar{d}_{\mu\nu\sigma}{}^\rho = \mathring{d}_{\mu\nu\sigma}{}^\rho, \\
\overline{\partial_0 d_{\mu\nu\sigma}{}^\rho} = \mathring{D}_{\mu\nu\sigma}{}^\rho, \quad \bar{\Theta} = \mathring{\Theta} = 0, \quad \overline{\partial_0 \Theta} = \mathring{\Omega}, \quad \bar{s} = \mathring{s} = 0, \quad \overline{\partial_0 s} = \mathring{S})
\end{aligned}$$

where $\mathring{g}_{ij} = h_{ij}$ and the trace- and divergence-free part of $\mathring{d}_{0i0j} = D_{ij}$ are the free data, is a solution of the MCFE (2.1)-(2.6) in the

$$(R = 0, \bar{s} = 0, \bar{g}_{00} = -1, \bar{g}_{0i} = 0, \hat{g}_{\mu\nu} = \mathring{g}_{\mu\nu})\text{-wave-map gauge}$$

if and only if the initial data have their usual algebraic properties and solve the constraint equations (A.8)-(A.15). The function Θ is positive in some neighborhood to the future of \mathcal{I}^- , and $d\Theta \neq 0$ on \mathcal{I}^- .

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